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MOTIVATOR'S SPECTRUM

AMAN BANSAL
JEE AIR 1st RANK

KUNAL GOYAL
JEE AIR 3rd RANK

METICULOUS ANALYSIS

PROPERTIES OF DEFINITE INTEGRAL

MATHS GALAXY
EXPONENTIAL AND LOGARITHMIC FUNCTIONS
CONDITIONAL PROBABILITY

CONCEPT MAP
COMPLEX NUMBERS

TOUCH UPS
BINOMIAL THEOREM AND LINEAR INEQUALITIES
APPLICATION OF DERIVATIVE

JEE ADVANCED DRILL
DIFFERENTIATION

SOLVED PAPER
EAMCET (AP)
ENGG. ENTRANCE EXAM

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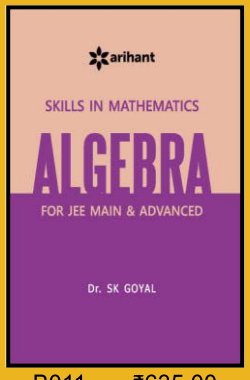
TEXT BOOKS OF MATHEMATICS FOR JEE MAIN & ADVANCED

BY DR. SK GOYAL & AMIT M AGARWAL



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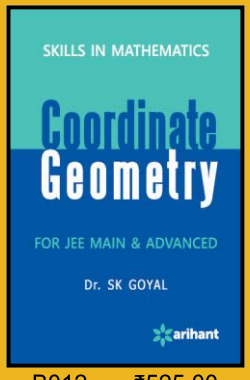

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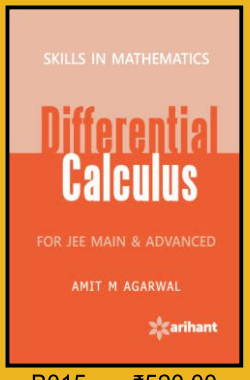

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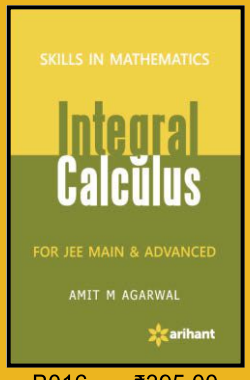

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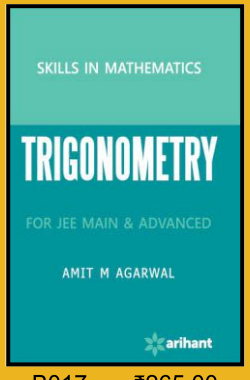

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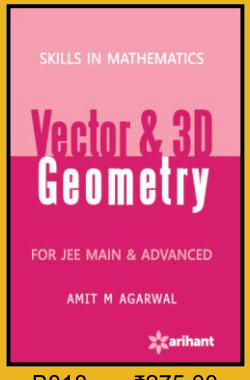

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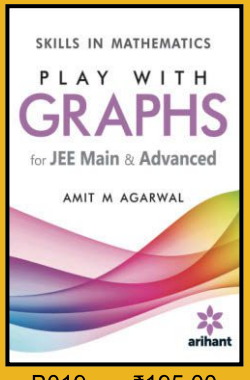

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Issue 9 **Vol 2**

Dear Aspirants,

By now your studies would be in full swing after such a valuable summer break. The spectrum seed which was sown few years back is also taking the shape of a tree, gradually with multiple branches. The overwhelming response given by you plays a vital role in its growth.

Perhaps the most central idea in Mathematics is the notion of a function. The word function was apparently introduced by Rene Descartes in 1637. For him, a function simply meant any positive integral power of a variable x. Gottfried Wilhelm von Leibniz used this word to denote any quantity associated with a curve. However it was Leonhard Euler, who employed the word to any mean any equation or formula involving variables and constants. It is Euler's idea we most often use today in Mathematics. I consider mathematics as one of the foundation stones of science. It is one of the oldest basic tools on which different branches of science have emerged. To me Mathematics has romance too. You can also feel the same after going through the following lines of **William Blake**

"To see the world in a grain of sand,
And a heaven in a wild flower;
Hold infinity in the palm of your hand,
And eternity in an hour"

It is infinity which creates such a romance. In this infinite world every time on exploration some examples emerge in front of us as inspirational stones. Krishna Kumar's story also has same feel and touch.

We are trying hard to provide more and more material for your practice within the purview of our limitations. Hope you all are also trying best to Crack the hard nut called examination. The spectrum team will be anxiously waiting for your feed backs and comments.

Good luck

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JEE TOPPER

AMAN BANSAL

AIR 1st RANK

Q Achieving top position was a surprise for you or you were expecting it.
Aman- It was a pleasant surprise, I was expecting a good rank but not the top most.

Q Please share your feelings after achieving such a mile stone.
Aman- Feeling very good and relaxed.

Q In how many attempts did you get this success?
Aman- Ist attempt

Q How did you manage to prepare for board examinations and competitive exams simultaneously?
Aman- The syllabus for JEE and board exams is almost same but I have put some extra efforts for Board exams.

Q What was your subject wise score in 12th boards?
Aman- 96.5%

Q Please share the secret of your success with our readers.
Aman- The student must clear all his doubts and should have discussions regarding major problems among friends, it will help them to overcome the problem of doubts.

Q Tell us something about your source of inspiration.
Aman- I was greatly inspired by the success of Ist RANKER Mr. Chitrang in JEE ADVANCED 2014. After meeting with Mr. Chitrang , I have decided to get a good rank in JEE ADVANCED.

MESSAGE TO FUTURE ASPIRANTS

Always focus on your weak areas, try to overcome regular mistakes, Revise as much as possible, practice the previous year papers. Don't waste your time, focus on your goal.

Q What was your study plan; please explain in details so that our readers can also follow it?

Aman- Put maximum focus on Maths and covering the topics of Physics and Chemistry day by day.

Q Did your study plan remain same throughout the year; at the start of the session, after completion of syllabus and just 1-2 months before the exam or you had different plans for different phases?

Aman-I have started the preparation in 11th and after completion of syllabus, put focus on the practice only. After Board exams, one must solve last year solved papers like 38 years chapterwise papers of JEE ADVANCED to get the pulse of this competition.

Q How many hours of study, is sufficient for such a success?

Aman- A regular study of 7 to 9 hours is required, if a student is doing self study. If anyone is attending Coaching classes, then he/she must study another 3 to 4 hours without any disturbance to understand the subject studied in coaching.

Q Tell us something about your study techniques like how do you memorize the facts, how did you have your revision and how you assessed your preparation from time to time?

Aman- I use to underline my common mistakes and use to repeat any fact 2 to 3 times to memorize.

Q Tell us about your family and the role of your family members in guiding you towards this spectacular performance in the exam?

Aman- My parents and sister are very helpful for me but my grandmother is the real source of inspiration for getting this position.

Q Share something about the Weak & Strong points of your personality.

Aman- Maths and Inorganic Chemistry were the weak points and Physics , Organic and Physical Chemistry are strong areas.

Q Did you devote equal time to all the subjects or more weightage given to one or two subjects as compared to other subjects?

Aman- Maximum time to Maths and equal weightage to Physics and Chemistry.

Q Were there any weak areas of yours in any subject(s)? How did you cope up with those?

Aman- Maths and Inorganic Chemistry , I have put maximum focus on these two and practiced it very hard.

Q How your teachers/mentors helped in achieving this goal?

Aman- Teachers/mentors always motivated me and keep me away from the nervousness always . Their expert notes and tips proved very useful for me.

Q Which is your dream institution and which stream do you wish to pursue?

Aman- IIT Mumbai , Computer Science.

Q Besides studies, what do you do, i.e. what are your hobbies and how you match up with these hobbies during your studies? What you do during the breaks from your studies?

Aman- Badminton and other Indoor games.

Q Now-a-days social media like Facebook, Twitter etc is a big trend? Do you also engage with these, does it affect the studies? Please give some advice to our readers.

Aman- I kept myself away from Social media, it is good if someone uses it as a tool for his studies.

JEE TOPPER

KUNAL GOYAL

AIR 3rd RANK

Q Achieving top position was a surprise for you or you were expecting it.
Kunal- It was a surprise for me , I was expecting a rank around 100 but coming at No. 3 is really a pleasant surprise.

Q Please share your feelings after achieving such a mile stone.
Kunal- Feeling too good and very confident.

Q In how many attempts did you get this success?
Kunal- Ist attempt

Q How did you manage to prepare for board examinations and competitive exams simultaneously?
Kunal- Started preparing for JEE from 11th class and gave last two months for Board exams.

Q What were your subject wise score in 12th boards?
Kunal- 95.2 %

Q Please share the secret of your success with our readers.
Kunal- One should focus on his goal and studies and should not compare himself with others.

Q Tell us something about your source of inspiration.
Kunal- Inspired by our Missile man Dr. APJ ABDUL KALAAM AZAD.

MESSAGE TO FUTURE ASPIRANTS

Don't take much stress on yourself, highlight main points, make short notes, don't take pressure of rank on yourself.

Q What was your study plan; please explain in details so that our readers can also follow it?

Kunal-After coming from the school and coaching , I use to revise all the chapters taught on that day.

Q Did your study plan remain same throughout the year; at the start of the session, after completion of syllabus and just 1-2 months before the exam or you had different plans for different phases?

Kunal- I have started the preparation for JEE in 11 th and use to study for Boards and JEE together.

Q How many hours of study, is sufficient for such a success?

Kunal- A regular study of 5 to 6 hours is required.

Q Tell us something about your study techniques like how do you memorize the facts, how did you have your revision and how you assessed your preparation from time to time?

Kunal- Make short notes , Revise and practice regularly.

Q Tell us about your family and the role of your family members in guiding you towards this spectacular performance in the exam.

Kunal- My parents are real source of inspiration they always motivated me always , there was no pressure from their side to get any rank.

Q Share something about the Weak & Strong points of your personality.

Kunal- I use to solve difficult questions with ease but use to do silly mistakes in easy questions.

Q Did you devote equal time to all the subjects or more weightage given to one or two subjects as compared to other subjects?

Kunal- Maximum time to Maths and equal weightage to Physics and Chemistry.

Q Were there any weak areas of yours in any subject(s)? How did you cope up with those?

Kunal- Waves in Physics , Inorganic Chemistry , Indefinite Integers in Maths. I use to do maximum practice of these to overcome it.

Q How your teachers/mentors helped in achieving this goal?

Kunal- Teachers/mentors always motivated me and books suggested by them proved very useful.

Q Which is your dream institution and which stream do you wish to pursue?

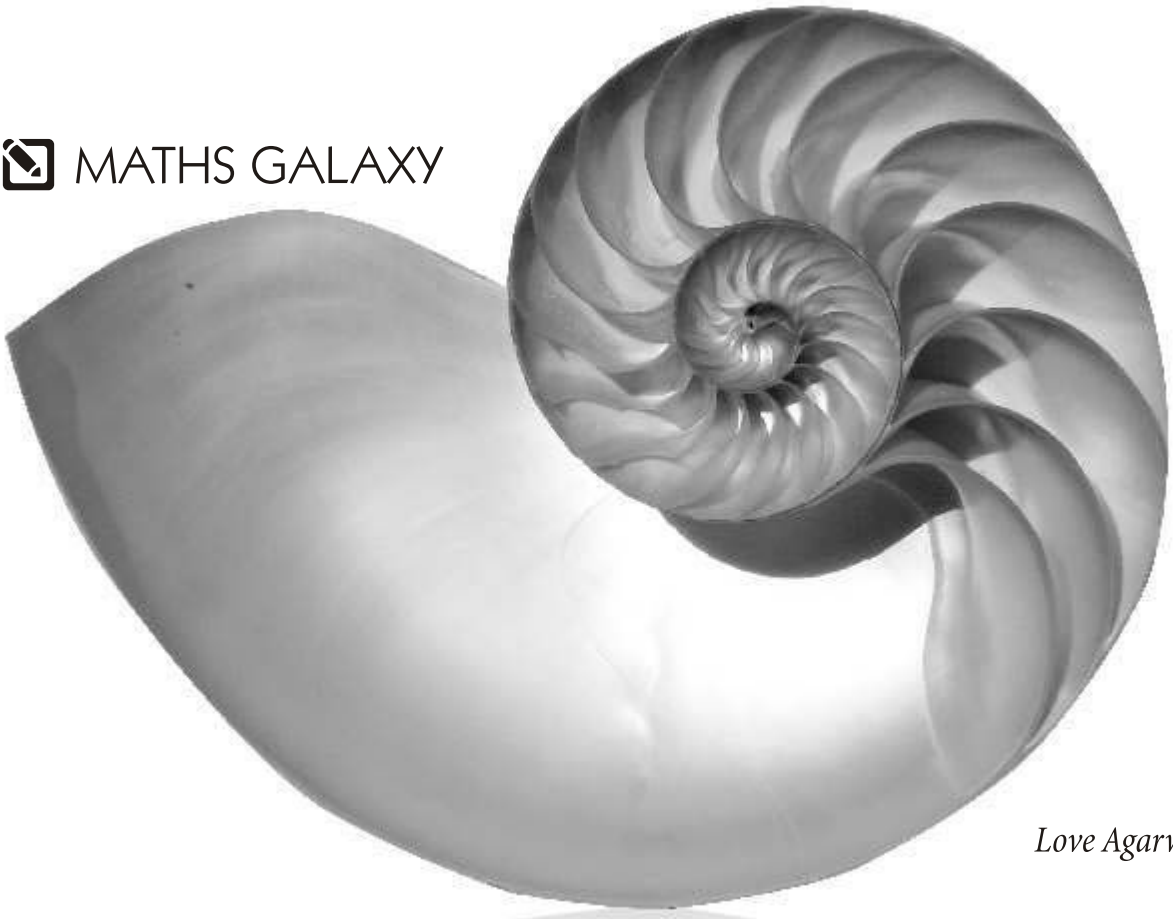
Kunal- IIT Mumbai , Computer Science.

Q Besides studies, what do you do, i.e. what are your hobbies and how you match up with these hobbies during your studies? What you do during the breaks from your studies?

Kunal- Basketball and swimming.

Q Now-a-days social media like Facebook, Twitter etc is a big trend. Do you also engage with these, does it affect the studies? Please give some advice to our readers.

Kunal- I used Whatsapp for discussing questions and problems among friend circle.



Love Agarwal

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

EXPONENTIAL AND LOGARITHMIC FUNCTIONS ARE VERY IMPORTANT TOPICS IN MATHEMATICS. LOGARITHM IS USED IN SOLVING COMPLEX CALCULATIONS, PROGRAMMING OF CALCULATORS ETC. EXPONENTIAL FUNCTION HAS WIDE USE IN CALCULATIONS OF INTEREST IN INVESTMENT, GROWTH ETC.

Exponential Function

Let $y = f(x) = a^x$

where $a > 0$; $a \neq 1$ and $x \in R$.

Here, $a^x > 0$, $\forall\ x \in R$. Thus, the range of the function is $(0, \infty)$.

Exponential functions always have some positive number other than 1 as the base.

For negative base it may be not defined. For example $(-1)^{1/2}$ is not defined. Also, we do not take base as 0 and 1.

POINTS TO REMEMBER

(i) $1^x = 1$ (ii) $a^1 = a$

(iii) $a^0 = 1$ (iv) $a^{-x} = \frac{1}{a^x}$

(v) $(a^x)^y = a^{xy}$ (vi) $a^x \cdot a^y = a^{x+y}$

(vii) $\frac{a^x}{a^y} = a^{x-y}$ (viii) $\sqrt[y]{a^x} = a^{x/y}$

(ix) $a^x = a^y \Rightarrow x = y$ (x) $a^x = b^x \Rightarrow$ either $a = b$ or $x = 0$

Example 1 If $4^x + 2^{2x-1} = 3^{x+\frac{1}{2}} + 3^{x-\frac{1}{2}}$, then x is equal to

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) 1

Sol. ***(b)*** We have,

$$4^x + 2^{2x-1} = 3^{x+\frac{1}{2}} + 3^{x-\frac{1}{2}}$$

$$\Rightarrow \quad 2^1 \cdot 2^{2x-1} + 2^{2x-1} = 3^{x-\frac{1}{2}} \times 3^1 + 3^{x-\frac{1}{2}}$$

$$\Rightarrow \quad 2^{2x-1}(2+1) = 3^{x-\frac{1}{2}}(3+1)$$

$$\Rightarrow \quad 2^{2x-1} \times 3 = 3^{x-\frac{1}{2}} \times 4$$

$$\Rightarrow \quad 2^{2x-3} = 3^{x-\frac{3}{2}}$$

$$\Rightarrow \quad (2^2)^{x-\frac{3}{2}} = 3^{x-\frac{3}{2}}$$

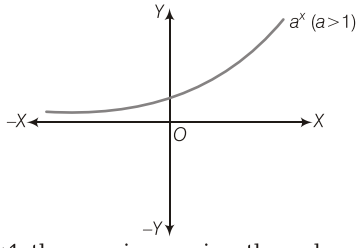
$$\Rightarrow \quad 4^{x-\frac{3}{2}} = 3^{x-\frac{3}{2}}$$

$$\Rightarrow \quad x - \frac{3}{2} = 0 \qquad [\because 4 \neq 3]$$

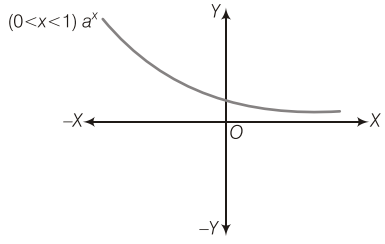
$$\Rightarrow \quad x = \frac{3}{2}$$

Graphs of Exponential Function

When $a > 1$, then on increasing the value of x, value of a^x also increases.



When $0 < a < 1$, then on increasing the value of x, value of a^x decreases.



POINTS TO REMEMBER

(i) $f(x) = a^x$, $0 < a < 1$ is a decreasing function. If $x_2 > x_1$, then $a^{x_2} < a^{x_1}$

(ii) $f(x) = a^x$, $a > 1$ is an increasing function. If $x_2 > x_1$, then $a^{x_2} > a^{x_1}$

Logarithmic Function

If a is a positive real number, other than 1 and x is a rational number such that $a^x = N$, then we say that logarithm of N to the base a is x or x is the logarithm of N to the base a, written as $\log_a N = x$

Thus, $\quad a^x = N \quad \Leftrightarrow \quad \log_a N = x$

Fundamental Laws of Logarithms

Following are the fundamental laws of logarithms to any base a, where $a > 0$ and $a \neq 1$.

- If $m, n > 0$, then $\log_a (mn) = \log_a m + \log_a n$
- If $m, n > 0$, then $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$
- If $m, n > 0$, then $\log_a (m^n) = n \log_a m$
- $\log_a 1 = 0$, i.e. the log of 1 to any base is always zero.
- $\log_a a = 1$, i.e. the log of any positive quantity of the same base is always one.
- If m is positive rational number and a, b are positive real numbers such that $a \neq 1$, $b \neq 1$, then
$$\log_a m = \frac{\log_a m}{\log_b m} \text{ or } \log_a m \times \log_b a = \log_b m$$
- If a is a positive real number and n is a positive rational number, then $a^{\log_a n} = n$
- If a is a positive real number and n is a positive rational number, then $\log_{a^q} n^p = \frac{p}{q} \log_a n$
- If x, y and a $(\neq 1)$ are positive real numbers, then $x^{\log_a y} = y^{\log_a x}$

Example 2 The value of

$$6 + \log_{3/2}\left\{\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\cdots}}}\right\} \text{ is}$$

(a) 1 (b) 2 (c) 3 (d) 4 **[IIT JEE 2012]**

Sol. ***(d)*** Let

$$\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\cdots}}} = y$$

Then, $\quad \sqrt{4 - \frac{1}{3\sqrt{2}}y} = y$

$$\Rightarrow \quad 4 - \frac{1}{3\sqrt{2}}y = y^2$$

$$\Rightarrow \quad y^2 + \frac{1}{3\sqrt{2}}y - 4 = 0$$

$$\Rightarrow \quad 3\sqrt{2}y^2 + y - 12\sqrt{2} = 0$$

$$\Rightarrow \quad y = \frac{-1 \pm \sqrt{1 - 4(3\sqrt{2})(-12\sqrt{2})}}{2 \times 3\sqrt{2}}$$

$$\Rightarrow \quad y = \frac{-1 \pm 17}{6\sqrt{2}}$$

$\Rightarrow \quad y = \frac{16}{6\sqrt{2}}, \frac{-18}{6\sqrt{2}}$

$\Rightarrow \quad y = \frac{8}{3\sqrt{2}}, -\frac{3}{\sqrt{2}}$

$\Rightarrow \quad y = \frac{8}{3\sqrt{2}} \qquad \left[\because y \neq -\frac{3}{\sqrt{2}}\right]$

$\therefore$ Required value $= 6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}} \times \frac{8}{3\sqrt{2}}\right)$

$= 6 + \log_{3/2}\left(\frac{4}{9}\right) = 6 + \log_{3/2}\left(\frac{3}{2}\right)^{-2}$

$= 6 - 2\log_{3/2}\left(\frac{3}{2}\right) = 6 - 2 = 4$

Example 3 If $\frac{\log_{10} a}{2} = \frac{\log_{10} b}{3} = \frac{\log_{10} c}{5}$, then bc is equal to

(a) a *(b)* a^2 *(c)* a^3 *(d)* a^4

Sol. ***(d)*** We have,

$\frac{\log_{10} a}{2} = \frac{\log_{10} b}{3} = \frac{\log_{10} c}{5} = \lambda$ (say)

$\Rightarrow \quad a = 10^{2\lambda}, b = 10^{3\lambda}, c = 10^{5\lambda}$

$\Rightarrow \quad bc = 10^{8\lambda} = (10^{2\lambda})^4 = a^4$

Example 4 If $y = \log_{4-x}(x-1)$, then x lies in the interval

(a) (3, 4) *(b)* (1, 4) *(c)* (1, 3) $\cup$ (3, 4) *(d)* (1, 3)

Sol. ***(c)*** As we know that for $f(x) = \log_a x$,

$a > 0, a \neq 0$ and $x > 0$

$\therefore \quad 4 - x > 0$ and $4 - x \neq 1$

Also, $\quad x - 1 > 0 \Rightarrow x < 4, x \neq 3, x > 1$

$\therefore \quad x \in (1, 3) \cup (3, 4)$

Example 5 If a, b, c are positive real numbers, then $a^{\log b - \log c} \times b^{\log c - \log a} \times c^{\log a - \log b}$ is equal to

(a) 0 *(b)* 1 *(c)* -1 *(d)* None of these

Sol. ***(b)*** We have,

$\log\{a^{\log b - \log c} \cdot b^{\log c - \log a} \cdot c^{\log a - \log b}\}$

$= (\log b - \log c)\log a + (\log c - \log a)\log b + (\log a - \log b)\log c$

$= 0$

$\therefore \quad a^{\log b - \log c} \cdot b^{\log c - \log a} \cdot c^{\log a - \log b} = 1$

Example 6 If $\frac{1}{\log_2 a} + \frac{1}{\log_4 a} + \frac{1}{\log_8 a} + \frac{1}{\log_{16} a} + \ldots + \frac{1}{\log_{2^n} a} = \frac{n(n+1)}{\lambda}$, then λ equals

(a) $\log_2 a$ *(b)* $\log_a 4$ *(c)* $\log_2 a^2$ *(d)* $\log_4 a$

Sol. ***(c)*** We have,

$\frac{1}{\log_2 a} + \frac{1}{\log_4 a} + \frac{1}{\log_8 a} + \frac{1}{\log_{16} a} + \ldots + \frac{1}{\log_{2^n} a} = \frac{n(n+1)}{\lambda}$

$\Rightarrow \quad \log_a 2 + \log_a 2^2 + \log_a 2^3 + \ldots + \log_a 2^n = \frac{n(n+1)}{\lambda}$

$\Rightarrow \quad \log_a 2 + 2\log_a 2 + 3\log_a 2 + \ldots + n\log_a 2 = \frac{n(n+1)}{\lambda}$

$\Rightarrow \quad (1 + 2 + 3 + \ldots + n)\log_a 2 = \frac{n(n+1)}{\lambda}$

$\Rightarrow \quad \frac{n(n+1)}{2}\log_a 2 = \frac{n(n+1)}{\lambda}$

$\Rightarrow \quad \frac{1}{2}\log_a 2 = \frac{1}{\lambda}$

$\Rightarrow \quad \lambda = 2\log_2 a$

$\Rightarrow \quad \lambda = \log_2 a^2$

Example 7 If $3^x = 4^{x-1}$, then x cannot be equal to [IIT JEE 2013]

(a) $\frac{2\log_3 2}{2\log_3 2 - 1}$ *(b)* $\frac{2}{2 - \log_2 3}$

(c) $\frac{1}{1 - \log_4 3}$ *(d)* $\frac{2\log_2 3}{2\log_2 3 - 1}$

Sol. ***(d)*** We have,

$3^x = 4^{x-1}$

$\Rightarrow \quad x\log_{10} 3 = (x-1)\log_{10} 4$

$\Rightarrow \quad x = (x-1)\log_3 4$

$\Rightarrow \quad x = 2(x-1)\log_3 2$

$\Rightarrow x(2\log_3 2 - 1) = 2\log_3 2$

$\Rightarrow \quad x = \frac{2\log_3 2}{2\log_3 2 - 1}$

Now, $\quad x = \frac{2\log_3 2}{2\log_3 2 - 1}$

$\Rightarrow \quad x = \frac{2}{2 - \frac{1}{\log_3 2}} = \frac{2}{2 - \log_2 3}$

$\Rightarrow \quad x = \frac{1}{1 - \frac{1}{2}\log_2 3} = \frac{1}{1 - \log_{2^2} 3} = \frac{1}{1 - \log_4 3}$

Hence, options (a), (b) and (c) are correct and option (d) is not correct.

Some Useful Results

Following are some useful results on logarithms

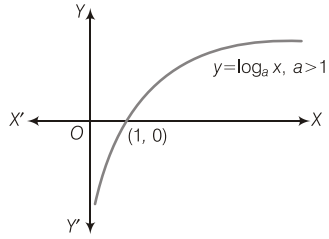
1. If $a > 1$, then

(i) $\log_a x < 0$ for all x satisfying $0 < x < 1$

(ii) $\log_a x = 0$ for $x = 1$

(iii) $\log_a x > 0$ for $x > 1$

(iv) $x > y \Leftrightarrow \log_a x > \log_a y$, i.e. $\log_a x$ is an increasing function.



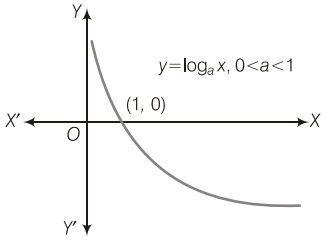
2. If $0 < a < 1$, then

(i) $\log_a x < 0$ for all $x > 1$

(ii) $\log_a x = 0$ for $x = 1$

(iii) $\log_a x > 0$ for all x satisfying $0 < x < 1$

(iv) $x > y \Leftrightarrow \log_a x < \log_a y$, i.e. $\log_a x$ is a decreasing function.



3. If $a > 1$, then

(i) $\log_a x > 1$, if $x > a$ (ii) $\log_a x < 1$, if $0 < x < a$

(iii) $\log_a x = 1$, if $x = a$

4. If $0 < a < 1$, then

(i) $\log_a x > 1$, if $0 < x < a$ (ii) $\log_a x < 1$, if $x > a$

(iii) $\log_a x = 1$, if $x = a$

5. (i) $\log_a x > 1 \Rightarrow \begin{cases} x > a, & \text{if } a > 1 \\ 0 < x < a, & \text{if } 0 < a < 1 \end{cases}$

(ii) $\log_a x < 1 \Rightarrow \begin{cases} 0 < x < a, & \text{if } a > 1 \\ x > a, & \text{if } 0 < a < 1 \end{cases}$

Example 8 If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval [IIT JEE 2007]

(a) $(2, \infty)$ *(b)* $(-2, -1)$

(c) $(1, 2)$ *(d)* $(-\infty, 2)$

Sol. ***(a)*** We have, $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

Clearly, it is defined for $x > 1$.

Now, $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

$\Rightarrow \log_{0.3}(x-1) < \log_{(0.3)^2}(x-1)$

$\Rightarrow \log_{0.3}(x-1) < \frac{1}{2}\log_{0.3}(x-1)$

$\Rightarrow \frac{1}{2}\log_{0.3}(x-1) < 0$

$\Rightarrow \log_{0.3}(x-1) < 0$

$\Rightarrow x - 1 > (0.3)^0$

$\Rightarrow x - 1 > 1$

$\Rightarrow x > 2$

$\Rightarrow x \in (2, \infty)$

Example 9 The number of values of x satisfying $1 + \log_5(x^2+1) \geq \log_5(x^2+4x+1)$, is

(a) 1

(b) 2

(c) 3

(d) infinitely many

Sol. ***(d)*** We have,

$1 + \log_5(x^2+1) \geq \log_5(x^2+4x+1)$

$\Rightarrow \log_5 5 + \log_5(x^2+1) \geq \log_5(x^2+4x+1)$

$\Rightarrow 5(x^2+1) \geq x^2 + 4x + 1$

$\Rightarrow 4x^2 - 4x + 4 \geq 0$

$\Rightarrow x^2 - x + 1 \geq 0$, which is true for all $x \in R$.

Also, $\log_5(x^2+4x+1)$ is defined for $x^2 + 4x + 1 > 0$

i.e. $x \in (-\infty, -2-\sqrt{3}) \cup (-2+\sqrt{3}, \infty)$

Hence, the solution set of the given inequation is

$(-\infty, -2-\sqrt{3}) \cup (-2+\sqrt{3}, \infty)$

Example 10 If $\log_{0.5}\sin x = 1 - \log_{0.5}\cos x$, then the number of solutions in the interval $[-2\pi, 2\pi]$ is

(a) 1 *(b)* 2. *(c)* 3 *(d)* 4

Sol. ***(b)*** Clearly, $\log_{0.5}\sin x$ and $\log_{0.5}\cos x$ are defined, if $x \in \left(0, \frac{\pi}{2}\right)$

Now, $\log_{0.5}\sin x = 1 - \log_{0.5}\cos x$

$\Rightarrow \log_{0.5}\sin x + \log_{0.5}\cos x = 1$

$\Rightarrow \log_{0.5}\sin x \cos x = 1$

$\Rightarrow \sin x \cos x = (0.5)^1$

$\Rightarrow \sin 2x = 1$

$\Rightarrow x = \frac{\pi}{4}, \frac{-3\pi}{4}$

INSPIRING ICONS

It is a fairy tale like story of courage, affection and truly inspirational. Two siblings from an impoverished Bihar farmer's family worked with determination and courage to clear IIT. Younger brother, Basant Kumar Pandit (18) carried his polio-afflicted elder brother, Krishna Kumar (20) on his back and both succeeded in JEE Advanced 2016.

Combating disability of legs following a childhood case of polio, Krishna has shown the world that courage can make you fly. However, Krishna's younger brother, Basant is credited equally for the success of Krishna, as he has been carrying the latter on his back since childhood when he was detected with polio and needed someone's support to go to school. Born to a farmer, Madan Pandit of Paroriya village in Samastipur region of Bihar, Krishna bagged AIR 38 in Physically Challenged category (PwD) category of JEE Advanced 2016 while Basant has got AIR 3,675 in Other Backward Caste (OBC) Category.

Smart Practice

1. If $10^x + 10^y = 10$, then domain is
(a) $x > 1$ (b) $x \in (0, 1)$ (c) $x < 1$ (d) $x \in (-\infty, \infty)$

2. If $0 \le a \le x$, then the minimum value of $\log_a x + \log_x x$ is
(a) 1 (b) 2 (c) 0 (d) None of these

3. The value of $\log ab - \log |b|$ is
(a) $\log a$ (b) $\log |a|$ (c) $-\log a$ (d) None of these

4. $x^{\log_5 x} > 5$ implies
(a) $x \in (0, \infty)$ (b) $x \in (0, 1/5) \cup (5, \infty)$
(c) $x \in (1, \infty)$ (d) $x \in (1, 2)$

5. If a, b, c are three consecutive positive integers, then $\log(1 + ca)$ is equal to
(a) $\log b$ (b) $\log\left(\frac{b}{2}\right)$ (c) $\log(2b)$ (d) $2\log b$

6. If $a(b-c)x^2 + b(c-a)xy + c(a-b)y^2 = 0$ is perfect square, then $\frac{\log(a+c) + \log(a-2b+c)}{\log(a-c)}$ is equal to
(a) 3 (b) 4 (c) 2 (d) 1

7. If $\log_{ax} x$, $\log_{bx} x$, $\log_{cx} x$ are in HP, where a, b, c, x belong to $(1, \infty)$, then a, b, c are in
(a) AP (b) GP (c) HP (d) None of these

8. The number of solutions of $\log_{\sin x}(2^{\tan x}) > 0$ in the interval $(0, \pi/2)$ is
(a) 0 (b) 1 (c) 2 (d) 3

9. The solution set of the equation $\log_x 2 \cdot \log_{2x} 2 = \log_{4x} 2$, is
(a) $\{2^{-\sqrt{2}}, 2^{\sqrt{2}}\}$ (b) $\left\{\frac{1}{2}, 2\right\}$ (c) $\left\{\frac{1}{4}, 4\right\}$ (d) None of these

10. If $n = 1999!$, then $\sum_{x=1}^{1999} \log_n x$ is equal to
(a) 1 (b) 0 (c) $\sqrt[1999]{1999}$ (d) -1

11. If $\log_{10}(x^3 + y^3) - \log_{10}(x^2 - xy + y^2) \le 2$, then the maximum value of xy, $\forall\, x \ge 0, y \ge 0$ is
(a) 2500 (b) 3000 (c) 1200 (d) 3500

12. The minimum value of $\frac{2^{\log_{2^{1/4}} x} - 3^{\log_{27}(x^2+1)^3} - 2x}{7^{4\log_{49} x} - x - 1}$,
$\forall\, x \in R$ is
(a) $\frac{3}{2}$ (b) $\frac{3}{4}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

13. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then
(a) $f(x_1) \cdot f(x_2) = f(x_1 + x_2)$ (b) $f(x+2) - 2f(x+1) + f(x) = 0$
(c) $f(x) + f(x+1) = f(x^2 + x)$ (d) $f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2}\right)$

14. If $S = \{x \in N : 2 + \log_{2^{1/2}} \sqrt{x+1} > 1 - \log_{1/2} \sqrt{4 - x^2}\}$, then
(a) $S = \{1\}$ (b) $S = Z$ (c) $S = N$ (d) None of these

15. The product of roots of the equation $\frac{\log_8(8/x^2)}{(\log_8 x)^2} = 3$ is
(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

16. If $a = (\log_2 5)^2$ and $b = (\log_2 20)$, then
(a) $a > b$ (b) $a < b$ (c) $a = b$ (d) None of these

17. If the equation $2^x + 4^y = 2^y + 4^x$ is solved for y in terms of x, where $x < 0$, then the sum of the solutions is
(a) $x \log_2(1 - 2^x)$ (b) $x + \log_2(1 - 2^x)$
(c) $\log_2(1 - 2^x)$ (d) $x \log_2(2^x + 1)$

18. The solution set of the inequation $\log_{(x+3)}(x^2 - x) < 1$ is
(a) (2, 3) (b) (– 2, 3) (c) (– 3, – 2) (d) (– 3, 2)

19. The number of solutions of the equation
$\log_2(x + 5) = 6 - x$ is/are
(a) 1 (b) 2 (c) 3 (d) 4

20. If $\frac{\log x}{2a + 3b - 5c} = \frac{\log y}{2b + 3c - 5a} = \frac{\log z}{2c + 3a - 5b}$, then xyz is equal to
(a) 2 (b) 1 (c) 0 (d) – 1

21. For the system of equation
$\log_{10}(x^3 - x^2) = \log_5 y^2$; $\log_{10}(y^3 - y^2) = \log_5 z^2$
$\log_{10}(z^3 - z^2) = \log_5 x^2$
Which of the following is/are true?
(a) there are infinite number of solutions
(b) there is unique solution with $x, y, z \in Q$
(c) there are exactly two solutions with $x, y, z \in Q$
(d) there is no solution

22. If $[\cdot]$ denotes the greatest integer function, then the value of natural number n satisfying the equation $[\log_2 1] + [\log_2 2] + [\log_2 3] + \ldots + [\log_2 n] = 1538$, is
(a) 255 (b) 256 (c) 254 (d) 313

23. The value of $e^{[\log_e x + \log_{\sqrt{e}} x + \log_{\sqrt[3]{e}} x + \log_{\sqrt[4]{e}} x + \ldots + \log_{\sqrt[10]{e}} x]}$ is
(a) x^{10} (b) e (c) x^{55} (d) 55^e

24. If $x > 1$, then the least value of $\log_2 x^3 - \log_x(0.125)$ is
(a) 1 (b) 2 (c) 4 (d) 6

25. **Statement I** $\log_{10} x < \log_\pi x < \log_e x < \log_2 x$
Statement II $x < y \Rightarrow \log_a x > \log_a y$, when $0 < a < 1$.
(a) Both Statements are correct
(b) Only Statement I is correct
(c) Only Statement II is correct
(d) Both Statements are incorrect

Answers with Explanation

1. ***(c)*** Given, $10^x + 10^y = 10$

$\Rightarrow \quad 10^y = 10 - 10^x$

$\Rightarrow \quad y = \log_{10}(10 - 10^x)$

$\Rightarrow \quad 10 - 10^x > 0 \Rightarrow 10 > 10^x$

$\Rightarrow \quad x < 1$

2. ***(b)*** We have,

$0 \le a \le x$

$\Rightarrow \quad \log_a x \ge 1$

$\Rightarrow \quad \log_a x + \log_x x \ge 1 + \log_x x$

$\Rightarrow \quad \log_a x + \log_x x \ge 1 + 1 = 2$

3. ***(b)*** $\log ab$ is defined, if $ab > 0$ or a and b have the same sign.

Case I $a, b > 0$

$\Rightarrow \quad \log ab - \log|b| = \log a + \log b - \log b$

$= \log a \quad \ldots(i)$

Case II $a, b < 0$

$\Rightarrow \quad \log ab - \log|b| = \log(-a) + \log(-b) - \log(-b)$

$= \log(-a) \quad \ldots(ii)$

From Eqs. (i) and (ii), we have

$\log ab - \log|b| = \log|a|$

4. ***(b)*** We have,

$x^{\log_5 x} > 5$

or $\log_5 x \log_5 x > 1$

$\Rightarrow \quad (\log_5 x - 1)(\log_5 x + 1) > 0$

$\Rightarrow \quad \log_5 x > 1$ or $\log_5 x < -1$

$\Rightarrow \quad x > 5$ or $x < \frac{1}{5}$

Also, we must have $x > 0$. Thus,

$x \in \left(0, \frac{1}{5}\right) \cup (5, \infty)$

5. ***(d)*** Since, a, b, c are three consecutive positive integers.

$\therefore \quad b = a + 1, c = a + 2$ and $2b = a + c$

Now, $\log(1 + ca)$

$= \log[1 + (a + 2)a]$

$= \log(a + 1)^2$

$= 2\log(a + 1)$

$= 2\log b$

6. ***(c)*** If $a(b - c)x^2 + b(c - a)xy + c(a - b)y^2 = 0$ is a perfect square, then

$b^2(c - a)^2 y^2 - 4ac(b - c)(a - b)y^2 = 0$

$\Rightarrow \quad b^2(c - a)^2 - 4ac(b - c)(a - b) = 0$

$\Rightarrow \quad b^2(c - a)^2 = 4ac(b - c)(a - b)$

$\Rightarrow \quad b^2(c + a)^2 = 4ac(ab - ac + bc)$

$\Rightarrow \quad \{b(c + a)\}^2 - 4abc(a + c) + 4a^2c^2 = 0$

$\Rightarrow \quad \{b(c + a) - 2ac\}^2 = 0$

$\Rightarrow \quad b = \frac{2ac}{a + c}$

Now, $\frac{\log(a + c) + \log(a - 2b + c)}{\log(a - c)}$

$= \frac{\log(a + c) + \log\left(a - \frac{4ac}{a + c} + c\right)}{\log(a - c)}$

$= \frac{\log(a - c)^2}{\log(a - c)} = 2$

7. ***(b)*** We have,

$\log_{ax} x, \log_{bx} x, \log_{cx} x$ are in HP.

$\Rightarrow \quad \log_x ax, \log_x bx, \log_x cx$ are in AP.

$\Rightarrow \quad \log_x a + 1, \log_x b + 1, \log_x c + 1$ are in AP.

$\Rightarrow \quad \log_x a, \log_x b, \log_x c$ are in AP.

$\Rightarrow \quad a, b, c$ are in GP.

8. ***(a)*** We have,

$\log_{\sin x}(2^{\tan x}) > 0$

$\Rightarrow \quad 2^{\tan x} < (\sin x)^0$

$\Rightarrow \quad 2^{\tan x} < 1$

But, $2^{\tan x} \not< 1$ for any $x \in \left(0, \frac{\pi}{2}\right)$.

Hence, the given inequation has no solution.

9. ***(a)*** We have,

$\log_x 2 \cdot \log_{2x} 2 = \log_{4x} 2$

$\Rightarrow \quad \frac{1}{\log_2 x} \cdot \frac{1}{\log_2 2x} = \frac{1}{\log_2 4x}, x \ne 1, x > 0$

$\Rightarrow \quad \frac{1}{\log_2 x} \cdot \frac{1}{1 + \log_2 x} = \frac{1}{2 + \log_2 x}$

$\Rightarrow \quad (\log_2 x + 2) = \log_2 x (1 + \log_2 x)$

$\Rightarrow \quad (\log_2 x)^2 = 2$

$\Rightarrow \quad \log_2 x = \pm\sqrt{2}$

$\Rightarrow \quad x = 2^{\sqrt{2}}, 2^{-\sqrt{2}}$

10. ***(a)*** We have,

$\sum_{x=1}^{1999} \log_n x$

$= \log_n 1 + \log_n 2 + \log_n 3 + \ldots + \log_n 1999$

$= \log_n (1 \cdot 2 \cdot 3 \ldots 1999)$

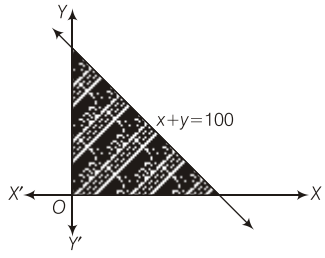
$= \log_n 1999!$

$= \log_n n = 1 \quad [\because n = 1999!]$

11. ***(a)*** We have,

$\log_{10}(x^3 + y^3) - \log_{10}(x^2 - xy + y^2) \le 2$

$\Rightarrow \quad \log_{10}(x + y) \le 2 \Rightarrow x + y \le 100$



It is given that $x \ge 0, y \ge 0$.

So, $x + y \le 100$, $x \ge 0$, $y \ge 0$ represents the triangular region in first quadrant. We know that, if $x + y$ is constant, then xy is maximum when $x = y$. Thus, xy is maximum when $x = y = 50$ and its maximum value is 2500.

12. ***(b)*** Let $y = \dfrac{2^{\log_{2^{1/4}} x} - 3^{\log_{27}(x^2+1)^3} - 2x}{7^{4\log_{49} x} - x - 1}$

$$= \frac{x^4 - (x^2 + 2x + 1)}{x^2 - x - 1}$$

$$= x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$\therefore$ Minimum value of y is $\dfrac{3}{4}$.

13. ***(d)*** $f(x_1) + f(x_2) = \log\left(\dfrac{1 + x_1}{1 - x_1}\right) + \log\left(\dfrac{1 + x_2}{1 - x_2}\right)$

$$= \log\left(\frac{1 + x_1}{1 - x_1} \cdot \frac{1 + x_2}{1 - x_2}\right)$$

$$= \log\left(\frac{1 + x_1x_2 + x_1 + x_2}{1 + x_1x_2 - x_1 - x_2}\right)$$

$$= \log\left(\frac{1 + \dfrac{x_1 + x_2}{1 + x_1x_2}}{1 - \dfrac{x_1 + x_2}{1 + x_1x_2}}\right) = f\left(\frac{x_1 + x_2}{1 + x_1x_2}\right)$$

14. ***(a)*** $2 + \log_{2^{1/2}} \sqrt{x + 1} > 1 - \log_{1/2} \sqrt{4 - x^2}$

$\Rightarrow \quad 1 + 2\log_2 \sqrt{x + 1} - \log_2 \sqrt{4 - x^2} > 0$

$\Rightarrow \quad \log_2 2 + \log_2 (x + 1) - \log_2 \sqrt{4 - x^2} > 0$

$\Rightarrow \quad \log_2 \dfrac{2(x + 1)}{\sqrt{4 - x^2}} > 0$

$\Rightarrow \quad \dfrac{2(x + 1)}{\sqrt{4 - x^2}} > 1$

$\Rightarrow \quad 4(x + 1)^2 > 4 - x^2$

$\Rightarrow \quad 4x^2 + 8x + 4 > 4 - x^2$

$\Rightarrow \quad 5x^2 + 8x > 0 \Rightarrow x > 0$ ...(i)

Also, $x + 1 > 0$ and $4 - x^2 > 0$

$\Rightarrow \quad x > -1$ and $-2 < x < 2$ ...(ii)

From Eqs. (i) and (ii), $0 < x < 2$

$\Rightarrow \quad x = 1$ as $x \in N$

15. ***(d)*** Let $\log_8 x = y$, then the given equation redues to

$$(1 - 2y) / y^2 = 3$$

$\Rightarrow \quad 3y^2 + 2y - 1 = 0$

$\Rightarrow \quad 3y^2 + 3y - y - 1 = 0$

$\Rightarrow \quad 3y(y + 1) - 1(y + 1) = 0$

$\Rightarrow \quad (y + 1)(3y - 1) = 0$

$\Rightarrow \quad y = \dfrac{1}{3}, -1$

$\Rightarrow \quad \log_8 x = \dfrac{1}{3}, -1$

$\Rightarrow \quad x = 2, \dfrac{1}{8}$

$\therefore$ Product of roots $= 2 \times \dfrac{1}{8} = \dfrac{1}{4}$

16. ***(a)*** $a - b = (\log_2 5)^2 - (\log_2 20) = (\log_2 5)^2 - (\log_2 5 + 2)$

Let $\log_2 5 = x$

Then, $a - b = x^2 - x - 2 = (x - 2)(x + 1)$

$= (\log_2 5 - 2)(\log_2 5 + 1)$

> 0

$\Rightarrow \quad a > b$

17. ***(b)*** We have, $2^x + 4^y = 2^y + 4^x$

$\Rightarrow \quad 2^{2y} - 2^y + 2^x(1 - 2^x) = 0$

Let $2^y = t$, we get

$t^2 - t + 2^x(1 - 2^x) = 0$

where, $t_1 = 2^{y_1}$ and $t_2 = 2^{y_2}$

$\therefore \quad t_1t_2 = 2^x(1 - 2^x)$

$\Rightarrow \quad 2^{y_1 + y_2} = 2^x(1 - 2^x)$

$\Rightarrow \quad y_1 + y_2 = x + \log_2(1 - 2^x)$

18. ***(c)*** We have,

$\log_{(x+3)}(x^2 - x) < 1$

$\Rightarrow \quad x(x - 1) > 0$

$\Rightarrow \quad x > 1$ or $x < 0$

If $x + 3 > 1 \Rightarrow x > -2$

Then, $x^2 - x < x + 3$

$\Rightarrow \quad x^2 - 2x - 3 < 0$

$\Rightarrow \quad (x - 3)(x + 1) < 0$

Hence, $x \in (-1, 0) \cup (1, 3)$

If $0 < (x + 3) < 1$

$\Rightarrow \quad -3 < x < -2$

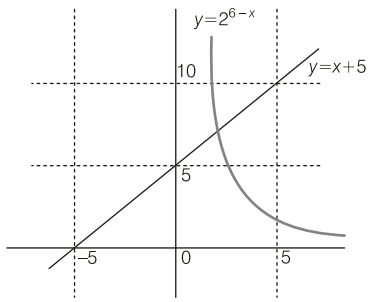
Then, $x^2 - x > x + 3$

$\Rightarrow \quad x^2 - 2x - 3 > 0$

$\Rightarrow \quad (x - 3)(x + 1) > 0$

$\Rightarrow \quad x \in (-3, -2)$

19. ***(a)*** Here, $x + 5 = 2^{6-x}$



Now, graph of $y = x + 5$ and $y = 2^{6-x}$ intersect only once.

Hence, there is only one solution.

20. ***(b)*** We have,

$$\frac{\log x}{2a + 3b - 5c} = \frac{\log y}{2b + 3c - 5a} = \frac{\log z}{2c + 3a - 5b}$$

$$\Rightarrow \quad \frac{\log x}{2a + 3b - 5c} = \frac{\log y}{2b + 3c - 5a} = \frac{\log z}{2c + 3a - 5b}$$

$$= \frac{\log x + \log y + \log z}{0}$$

$\Rightarrow \quad \log x + \log y + \log z = 0$

$\Rightarrow \quad \log(xyz) = 0$

$\Rightarrow \quad xyz = 1$

21. ***(b)*** The given equations are meaningful for $x, y, z > 1$.

Let $f(x) = 5^{\log_{10}(x^3 - x^2)}$.

Then, the given equations are

$$f(x) = y^2,\ f(y) = z^2$$

and $f(z) = x^2$

For $x > 1$, we find that $x^3 - x^2$ is an increasing function.

So, $f(x)$ is an increasing function.

Let $x > y$. Then,

$$f(x) > f(y) \quad [\because f \text{ is increasing}]$$

$$\Rightarrow \quad y^2 > z^2$$

$$\Rightarrow \quad y > z$$

$$\therefore \quad x > z$$

But, $y > z$

$$\Rightarrow \quad f(y) > f(z) \quad [\because f \text{ is increasing}]$$

$$\Rightarrow \quad z^2 > x^2$$

$\Rightarrow \quad z > x$, which is a contradiction.

$\therefore \quad x \not> y$

Similarly, $x < y$ leads us to a contradiction.

Therefore, $x \not< y$

Hence, $x = y$

Thus, we have $x = y = z$

Now, $\log_{10}(x^3 - x^2) = \log_5 y^2$

$\Rightarrow \quad \log_{10}(x^3 - x^2) = \log_5 x^2$

Let $\log_5 x = t$

i.e. $x = 5^t$

Then, $\log_{10}(x^3 - x^2) = \log_5 x^2$

$\Rightarrow \quad \log_{10}(5^{3t} - 5^{2t}) = 2t$

$\Rightarrow \quad 5^{3t} - 5^{2t} = 10^{2t}$

$\Rightarrow \quad 5^t - 1 = 4^t$

$\Rightarrow \quad 5^t - 4^t = 1$

Now, $5^t - 4^t = 4^t\left[\left(\frac{5}{4}\right)^t - 1\right]$

$\Rightarrow \quad 5^t - 4^t$ is an increasing function of t.

$\therefore \quad 5^t - 4^t = 1$

$\Rightarrow \quad t = 1$

Putting $t = 1$ in $x = 5^t$, we obtain $x = 5$

Hence, $x = y = z = 5$ is the unique solution of the given system of equations.

22. ***(a)*** If $2^{r-1} \le k < 2^r$, then

$$r - 1 \le \log_2 k < r$$

$$\Rightarrow \quad [\log_2 k] = r - 1$$

We have, $[\log_2 1] + [\log_2 2] + [\log_2 3] + \ldots + [\log_2 n] = 1538$

$$\Rightarrow [\log_2 2^0] + \sum_{k=2}^{2^2-1}[\log_2 k] + \sum_{k=2^2}^{2^3-1}[\log_2 k] + \sum_{k=2^3}^{2^4-1}[\log_2 k] + \ldots = 1538$$

$$\Rightarrow 0 + \sum_{k=2}^{2^2-1} 1 + \sum_{k=2^2}^{2^3-1} 2 + \sum_{k=2^3}^{2^4-1} 3 + \sum_{k=2^4}^{2^5-1} 4 + \ldots = 1538$$

$$\Rightarrow 0 + 1\times 2 + 2\times 2^2 + 3\times 2^3 + 4\times 2^4 + \ldots = 1538$$

We observe that

$$0 + 1\times 2 + 2\times 2^2 + 3\times 2^3 + 4\times 2^4 + 5\times 2^5 + 6\times 2^6 + 7\times 2^7$$

$$= 0 + 2 + 8 + 24 + 64 + 160 + 384 + 896 = 1538$$

$$\therefore \quad n = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7$$

$$= 1\cdot\left(\frac{2^8 - 1}{2 - 1}\right)$$

$$= 255$$

23. ***(c)*** We have,

$$\log_e x + \log_{\sqrt{e}} x + \log_{\sqrt[3]{e}} x + \log_{\sqrt[4]{e}} x + \ldots + \log_{\sqrt[10]{e}} x$$

$$= (1 + 2 + 3 + 4 + \ldots + 10)\log_e x$$

$$= \frac{10(10+1)}{2}\log_e x = 55\log_e x = \log_e x^{55}$$

$$\therefore \quad e^{\log_e x + \log_{\sqrt{e}} x + \log_{\sqrt[3]{e}} x + \ldots + \log_{\sqrt[10]{e}} x} = e^{\log_e x^{55}} = x^{55}$$

24. ***(d)*** We know that, AM $\ge$ GM

Now, $\log_2 x^3 - \log_x (0.125)$

$$= 3\log_2 x - \log_x\left(\frac{1}{8}\right)$$

$$= 3\log_2 x - \log_x 2^{-3}$$

$$= 3(\log_2 x + \log_x 2)$$

$$> 3\times 2 \quad [\because \log_2 x + \log_x 2 > 2]$$

$$> 6$$

25. ***(a)*** When $x > 1$, we have

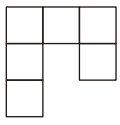
$$\log_x 10 > \log_x \pi > \log_x e > \log_x 2 \quad [\because 10 > \pi > e > 2]$$

$$\Rightarrow \quad \frac{1}{\log_{10} x} > \frac{1}{\log_\pi x} > \frac{1}{\log_e x} > \frac{1}{\log_2 x}$$

$$\Rightarrow \quad \log_{10} x < \log_\pi x < \log_e x < \log_2 x$$

Puzzling Maths

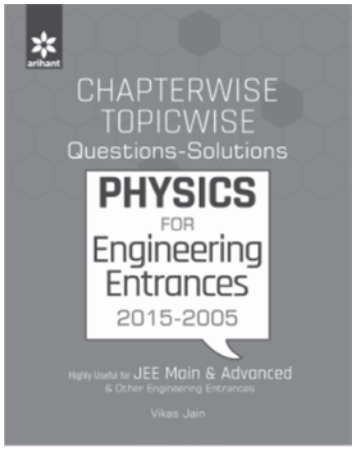
Determine all $m \times n$ rectangles that can be covered with hooks made up of 6 unit square, as in the figure



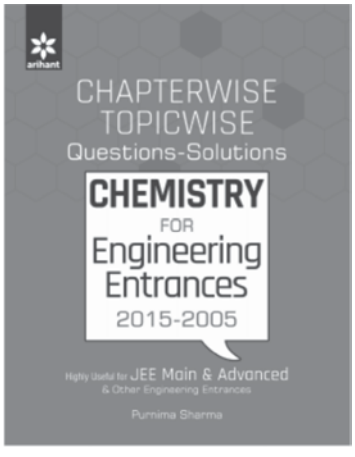
Rotations and reflections of hooks are allowed. The rectangle must be covered without gaps and overlaps. No part of a hook may cover area outside the rectangle.

CHAPTERWISE TOPICWISE SOLVED PAPERS

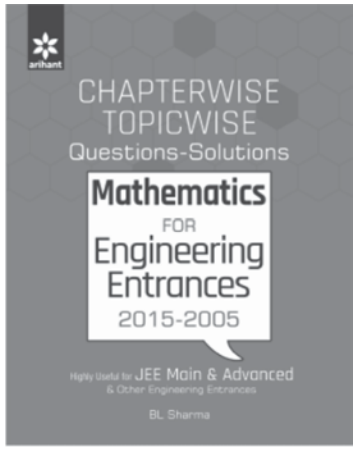
MASTER COLLECTION OF LAST 10 YEARS' QUESTIONS OF ENGINEERING ENTRANCES



B095 ₹665



B097 ₹665



C134 ₹665

The Highly Useful Resource that will Complement your Preparation as you can use these....

- With Daily Studies as Questions have been given Topicwise.
- For Revision Purpose after the Syllabus is completed.

arihant

EAMCET (AP) 2016

SOLVED PAPER

Engineering Entrance Examination

Held on 29 April, 2016

INSTRUCTION

- All questions are compulsory.
- Use of calculator is not permitted.
- The question paper consists of 80 questions.

EXAM CRUX

- In Maths paper, 46 questions were asked from Class XI and 34 questions from Class XII. Most of the questions were asked from all chapters but more weightage given to Algebra.
- According to topicwise distribution, the number of questions were asked from Class XI *Quadratic Equation* (4), *Binomial Theorem* and *Principle of Mathematic Induction* (4), *Complex Number* (4), *Permutation and Combination* (1), *Sets* (1), *Trigonometry* (8), *Statistics* (4), *Probability* (1), *Coordinate Geometry* (19).
- According to topicwise distribution, the number of questions were asked from Class XII–*Calculus* (22), *Matrix-Determinant* (3), *Inverse Trigonometric Function* (1), *Vector-3D* (7), *Probability* (1).
- As per difficulty level around 55% questions are easy, 32% are relatively moderate and rest 13% are tough.

1. The domain of the function $f(x) = \sqrt{\log_{0.5} x!}$ is

(a) {0, 1, 2, 3, …} *(b)* {1, 2, 3, …}
(c) (0, ∞) *(d)* {0, 1}

2. If $f(x) = |x-1| + |x-2| + |x-3|, 2 < x < 3$, then f is

(a) an onto function but not one-one
(b) a one-one function but not onto
(c) a bijection
(d) neither one-one nor onto

3. The greatest positive integer which divides $(n+16)(n+17)(n+18)(n+19)$, for all positive integers n, is

(a) 6 *(b)* 24 *(c)* 28 *(d)* 20

4. If a, b, c are distinct positive real numbers, then the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is

(a) < 0 *(b)* > 0 *(c)* 0 *(d)* ≥ 0

5. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in geometric progression with the same common ratio, then the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are

(a) vertices of an equilateral triangle
(b) vertices of a right angled triangle
(c) vertices of a right angled isosceles triangle
(d) collinear

6. The equations $x - y + 2z = 4$
$3x + y + 4z = 6$
$x + y + z = 1$ have

(a) unique solution *(b)* infinitely many solutions
(c) no solution *(d)* two solutions

7. The locus of the point representing the complex number z for which $|z+3|^2 - |z-3|^2 = 15$ is

(a) a circle *(b)* a parabola
(c) a straight line *(d)* an ellipse

8. $\frac{(1+i)^{2016}}{(1-i)^{2014}}$ is equal to

(a) $-2i$ *(b)* $2i$ *(c)* 2 *(d)* −2

9. If $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then the value of $|z_1 + z_2 + z_3|$ is

(a) 3 *(b)* 4
(c) 8 *(d)* 2

10. If $1, z_1, z_2, \ldots, z_{n-1}$ are the nth roots of unity, then $(1-z_1)(1-z_2)\ldots(1-z_{n-1})$ is equal to

(a) 0 *(b)* $n-1$
(c) n *(d)* 1

11. If $12^{4+2x^2} = (24\sqrt{3})^{3x^2-2}$, then x is equal to

(a) $\pm\sqrt{\frac{13}{12}}$ *(b)* $\pm\sqrt{\frac{14}{5}}$ *(c)* $\pm\sqrt{\frac{12}{13}}$ *(d)* $\pm\sqrt{\frac{5}{14}}$

12. The product and sum of the roots of the equation $|x|^2 - 5|x| - 24 = 0$ are respectively
(a) −64, 0 *(b)* −24, 5 *(c)* 5, −24 *(d)* 0, 72

13. The number of real roots of the equation $x^5 + 3x^3 + 4x + 30 = 0$ is
(a) 1 *(b)* 2 *(c)* 3 *(d)* 5

14. If the coefficients of the equation whose roots are k times the roots of the equation $x^3 + \frac{1}{4}x^2 - \frac{1}{16}x + \frac{1}{144} = 0$ are integers, then a possible value of k is
(a) 3 *(b)* 12
(c) 9 *(d)* 4

15. The sum of all 4-digit numbers that can be formed using the digits 2, 3, 4, 5, 6 without repetition, is
(a) 533820 *(b)* 532280
(c) 533280 *(d)* 532380

16. If a set A has 5 elements, then the number of ways of selecting two subsets P and Q from A such that P and Q are mutually disjoint, is
(a) 64 *(b)* 128
(c) 243 *(d)* 729

17. The coefficient of x^4 in the expansion of $(1 - x + x^2 - x^3)^4$ is
(a) 31 *(b)* 30
(c) 25 *(d)* −14

18. If the middle term in the expansion of $(1 + x)^{2n}$ is the greatest term, then x lies in the interval
(a) $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$ *(b)* $\left(\frac{n+1}{n}, \frac{n}{n+1}\right)$
(c) $(n-2, n)$ *(d)* $(n-1, n)$

19. To find the coefficient of x^4 in the expansion of $\frac{3x}{(x-2)(x-1)}$, the interval in which the expansion is valid, is
(a) $-2 < x < \infty$ *(b)* $-\frac{1}{2} < x < \frac{1}{2}$
(c) $-1 < x < 1$ *(d)* $-\infty < x < \infty$

20. If $(1 + \tan\alpha)(1 + \tan 4\alpha) = 2$, $\alpha \in \left(0, \frac{\pi}{16}\right)$ then α is equal to
(a) $\frac{\pi}{20}$ *(b)* $\frac{\pi}{30}$
(c) $\frac{\pi}{40}$ *(d)* $\frac{\pi}{60}$

21. If $\cos\theta = \frac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$, then one of the values of $\tan(\theta/2)$ is
(a) $\cot\frac{\beta}{2}\tan\frac{\alpha}{2}$ *(b)* $\tan\alpha\tan\frac{\beta}{2}$
(c) $\tan\frac{\beta}{2}\cot\frac{\alpha}{2}$ *(d)* $\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}$

22. The value of the expression
$$\frac{1 + \sin 2\alpha}{\cos(2\alpha - 2\pi)\tan\left(\alpha - \frac{3\pi}{4}\right)} - \frac{1}{4}\sin 2\alpha\left[\cot\frac{\alpha}{2} + \cot\left(\frac{3\pi}{2} + \frac{\alpha}{2}\right)\right] \text{ is}$$
(a) 0 *(b)* 1 *(c)* $\sin^2\frac{\alpha}{2}$ *(d)* $\sin^2\alpha$

23. If $(1/6)\sin\theta$, $\cos\theta$ and $\tan\theta$ are in geometric progression, then the solution set of θ is
(a) $2n\pi \pm \left(\frac{\pi}{6}\right)$ *(b)* $2n\pi \pm \left(\frac{\pi}{3}\right)$
(c) $n\pi + (-1)^n\left(\frac{\pi}{3}\right)$ *(d)* $n\pi + \left(\frac{\pi}{3}\right)$

24. If $x = \sin(2\tan^{-1}2)$ and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$ then
(a) $x > y$ *(b)* $x = y$ *(c)* $x = 0 = y$ *(d)* $x < y$

25. If $\cosh(x) = \frac{5}{4}$, then $\cosh(3x)$ is equal to
(a) $\frac{61}{16}$ *(b)* $\frac{63}{16}$ *(c)* $\frac{65}{16}$ *(d)* $\frac{61}{63}$

26. In ΔABC, if $x = \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$, $y = \tan\left(\frac{C-A}{2}\right)\tan\frac{B}{2}$ and $z = \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2}$, then $(x + y + z)$ is equal to
(a) xyz *(b)* $-xyz$ *(c)* $2xyz$ *(d)* $1/2\,xyz$

27. In ΔABC, if the sides a, b, c are in geometric progression and the largest angle exceeds the smallest angle by 60°, then $\cos B$ is equal to
(a) $\frac{\sqrt{13}+1}{4}$ *(b)* $\frac{1-\sqrt{13}}{4}$ *(c)* 1 *(d)* $\frac{\sqrt{13}-1}{4}$

28. In a ΔABC, if $\angle A = 90°$, then $\cos^{-1}\left(\frac{R}{r_2 + r_3}\right)$ is equal to
(a) 90° *(b)* 30° *(c)* 60° *(d)* 45°

29. The cartesian equation of the plane whose vector equation is
$\gamma = (1 + \lambda - \mu)\,\hat{\mathbf{i}} + (2 - \lambda)\,\hat{\mathbf{j}} + (3 - 2\lambda + 2\mu)\,\hat{\mathbf{k}}$, where λ, μ are scalars, is
(a) $2x + y = 5$ *(b)* $2x - y = 5$ *(c)* $2x - z = 5$ *(d)* $2x + z = 5$

30. For three vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, if $\mathbf{r} = 3\mathbf{p} + 4\mathbf{q}$ and $2\mathbf{r} = \mathbf{p} - 3\mathbf{q}$, then
(a) $|\mathbf{r}| < 2|\mathbf{q}|$ and $\mathbf{r}$, $\mathbf{q}$ have the same direction
(b) $|\mathbf{r}| > 2|\mathbf{q}|$ and $\mathbf{r}$, $\mathbf{q}$ have opposite directions
(c) $|\mathbf{r}| < 2|\mathbf{q}|$ and $\mathbf{r}$, $\mathbf{q}$ have opposite directions
(d) $|\mathbf{r}| > 2|\mathbf{q}|$ and $\mathbf{r}$, $\mathbf{q}$ have the same direction

31. If $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$, $\mathbf{b} = m\hat{\mathbf{i}} + n\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$ and $\mathbf{a} \times \mathbf{b} = 0$, then (m, n) is equal to
(a) $\left(\frac{-24}{5}, \frac{-36}{5}\right)$ *(b)* $\left(\frac{-24}{5}, \frac{36}{5}\right)$ *(c)* $\left(\frac{24}{5}, \frac{-36}{5}\right)$ *(d)* $\left(\frac{24}{5}, \frac{36}{5}\right)$

32. If $|\mathbf{a}| = 3, |\mathbf{b}| = 4$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 120°, then $|4\mathbf{a} + 3\mathbf{b}|$ is equal to

(a) 25 *(b)* 7 *(c)* 13 *(d)* 12

33. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-zero vectors such that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\,\mathbf{a}$, $\mathbf{c} \perp \mathbf{a}$ and θ is the angle between the vectors $\mathbf{b}$ and $\mathbf{c}$, then $\sin\theta$ is equal to

(a) $\frac{2\sqrt{2}}{3}$ *(b)* $\frac{1}{3}$ *(c)* $\frac{\sqrt{2}}{3}$ *(d)* $\frac{2}{3}$

34. If $a(\vec{\alpha} \times \vec{\beta}) + b(\vec{\beta} \times \vec{\gamma}) + c(\vec{\gamma} \times \vec{\alpha}) = 0$ and atleast one of the scalars a, b, c is non-zero, then the vectors $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are

(a) parallel *(b)* non-coplanar
(c) coplanar *(d)* mutually perpendicular

35. If the mean of 10 observations is 50 and the sum of the squares of the deviations of the observations from the mean is 250, then the coefficient of variation of those observations is

(a) 25 *(b)* 50
(c) 10 *(d)* 5

36. The variance of the first 50 even natural numbers is

(a) $\frac{833}{4}$ *(b)* 833 *(c)* 437 *(d)* $\frac{437}{4}$

37. 3 out of 6 vertices of a regular hexagon are chosen at a time at random. The probability that the triangle formed with these three vertices is an equilateral triangle, is

(a) $\frac{1}{2}$ *(b)* $\frac{1}{5}$ *(c)* $\frac{1}{10}$ *(d)* $\frac{1}{20}$

38. *A* speaks truth in 75% of the cases and *B* in 80% of the cases. Then, the probability that their statements about an incident do not match, is

(a) $\frac{7}{20}$ *(b)* $\frac{3}{20}$
(c) $\frac{2}{7}$ *(d)* $\frac{5}{7}$

39. If the mean and variance of a binomial distribution are 4 and 2 respectively, then the probability of 2 successes of that binomial variate *X*, is

(a) $\frac{1}{2}$ *(b)* $\frac{219}{256}$ *(c)* $\frac{37}{256}$ *(d)* $\frac{7}{64}$

40. In a city, 10 accidents take place in a span of 50 days. Assuming that the number of accidents follow the Poisson distribution, the probability that three or more accidents occur in a day, is

(a) $\sum_{k=3}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}, \lambda = 0.2$ *(b)* $\sum_{k=3}^{\infty} \frac{e^{\lambda}\lambda^k}{k!}, \lambda = 0.2$

(c) $1 - \sum_{k=3}^{3} \frac{e^{-\lambda}\lambda^k}{k!}, \lambda = 0.2$ *(d)* $\sum_{k=3}^{3} \frac{e^{-\lambda}\lambda^k}{k!}, \lambda = 0.2$

41. Equation of the locus of the centroid of the triangle whose vertices are $(a\cos k, a\sin k)$, $(b\sin k, -b\cos k)$ and $(1, 0)$, where k is a parameter, is

(a) $(1-3x)^2 + 9y^2 = a^2 + b^2$ *(b)* $(3x-1)^2 + 9y^2 = 2a^2 + 2b^2$
(c) $(3x+1)^2 + (3y)^2 = a^2 + b^2$ *(d)* $(3x+1)^2 + (3y)^2 = 3a^2 + 3b^2$

42. If the coordinate axes are rotated through an angle $\frac{\pi}{6}$ about the origin, then the transformed equation of $\sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0$ is

(a) $\sqrt{3}y^2 + xy = 0$ *(b)* $x^2 - y^2 = 0$
(c) $\sqrt{3}y^2 - xy = 0$ *(d)* $\sqrt{3}y^2 - 2xy = 0$

43. If the lines $x + 3y - 9 = 0$, $4x + by - 2 = 0$ and $2x - y - 4 = 0$ are concurrent, then the equation of the line passing through the point $(-5, 0)$ and concurrent with the given lines, is

(a) $2x + y + 10 = 0$ *(b)* $4x - 7y + 20 = 0$
(c) $x - y + 5 = 0$ *(d)* $x - 4y + 5 = 0$

44. The mid-point of the line segment joining the centroid and the orthocentre of the triangle whose vertices are $(a, b), (a, c)$ and (d, c), is

(a) $\left(\frac{5a+d}{6}, \frac{b+5c}{6}\right)$ *(b)* $\left(\frac{a+5d}{6}, \frac{5b+c}{6}\right)$
(c) (a, c) *(d)* $(0, 0)$

45. The distance from the origin to the image of $(1, 1)$ with respect to the line $x + y + 5 = 0$ is

(a) $7\sqrt{2}$ *(b)* $3\sqrt{2}$ *(c)* $6\sqrt{2}$ *(d)* $4\sqrt{2}$

46. The equation of the pair of lines joining the origin to the points of intersection of $x^2 + y^2 = 9$ and $x + y = 3$, is

(a) $x^2 + (3-y)^2 = 9$ *(b)* $(3+y)^2 + y^2 = 9$
(c) $x^2 - y^2 = 9$ *(d)* $xy = 0$

47. The orthocentre of the triangle formed by the lines $x + y = 1$ and $2y^2 - xy - 6x^2 = 0$ is

(a) $\left(\frac{4}{3}, \frac{4}{3}\right)$ *(b)* $\left(\frac{2}{3}, \frac{2}{3}\right)$ *(c)* $\left(\frac{2}{3}, \frac{-2}{3}\right)$ *(d)* $\left(\frac{4}{3}, \frac{-4}{3}\right)$

48. Let L be the line joining the origin to the point of intersection of the lines represented by $2x^2 - 3xy - 2y^2 + 10x + 5y = 0$. If L is perpendicular to the line $kx + y + 3 = 0$, then k is equal to

(a) $\frac{1}{2}$ *(b)* $-\frac{1}{2}$ *(c)* -1 *(d)* $\frac{1}{3}$

49. A circle $S = 0$ with radius $\sqrt{2}$ touches the line $x + y - 2 = 0$ at $(1, 1)$. Then, the length of the tangent drawn from the point $(1, 2)$ to $S = 0$ is

(a) 1 *(b)* $\sqrt{2}$ *(c)* $\sqrt{3}$ *(d)* 2

50. The normal drawn at $P(-1, 2)$ on the circle $x^2 + y^2 - 2x - 2y - 3 = 0$ meets the circle at another point Q. Then, the coordinates of Q are

(a) $(3, 0)$ *(b)* $(-3, 0)$ *(c)* $(2, 0)$ *(d)* $(-2, 0)$

51. If the lines $kx + 2y - 4 = 0$ and $5x - 2y - 4 = 0$ are conjugate with respect to the circle $x^2 + y^2 - 2x - 2y + 1 = 0$, then k is equal to

(a) 0 *(b)* 1 *(c)* 2 *(d)* 3

52. The angle between the tangents drawn from the origin to the circle $x^2 + y^2 + 4x - 6y + 4 = 0$ is

(a) $\tan^{-1}\left(\frac{5}{13}\right)$ *(b)* $\tan^{-1}\left(\frac{5}{12}\right)$ *(c)* $\tan^{-1}\left(\frac{12}{5}\right)$ *(d)* $\tan^{-1}\left(\frac{3}{5}\right)$

53. If the angle between the circles $x^2 + y^2 - 2x - 4y + c = 0$ and $x^2 + y^2 - 4x - 2y + 4 = 0$ is 60°, then c is equal to

(a) $\frac{3 \pm \sqrt{5}}{2}$ *(b)* $\frac{6 \pm \sqrt{5}}{2}$ *(c)* $\frac{9 \pm \sqrt{5}}{2}$ *(d)* $\frac{7 \pm \sqrt{5}}{2}$

54. A circle S cuts three circles $x^2 + y^2 - 4x - 2y + 4 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 + 4x + 2y + 1 = 0$ orthogonally. Then, the radius of S is

(a) $\sqrt{\frac{29}{8}}$ *(b)* $\sqrt{\frac{28}{11}}$ *(c)* $\sqrt{\frac{29}{7}}$ *(d)* $\sqrt{\frac{29}{5}}$

55. The distance between the vertex and the focus of the parabola $x^2 - 2x + 3y - 2 = 0$ is

(a) $\frac{4}{5}$ *(b)* $\frac{3}{4}$ *(c)* $\frac{1}{2}$ *(d)* $\frac{5}{6}$

56. If (x_1, y_1) and (x_2, y_2) are the end points of a focal chord of the parabola $y^2 = 5x$, then $4x_1x_2 + y_1y_2$ is equal to

(a) 25 *(b)* 5 *(c)* 0 *(d)* 5/4

57. The distance between the focii of the ellipse $x = 3\cos\theta$, $y = 4\sin\theta$ is

(a) $2\sqrt{7}$ *(b)* $7\sqrt{2}$ *(c)* $\sqrt{7}$ *(d)* $3\sqrt{7}$

58. The equations of the latusrectum of the ellipse $9x^2 + 25y^2 - 36x + 50y - 164 = 0$ are

(a) $x - 4 = 0, x + 2 = 0$ *(b)* $x - 6 = 0, x + 2 = 0$
(c) $x + 6 = 0, x - 2 = 0$ *(d)* $x + 4 = 0, x + 5 = 0$

59. The values of m for which the line $y = mx + 2$ becomes a tangent to the hyperbola $4x^2 - 9y^2 = 36$ is

(a) $\pm\frac{2}{3}$ *(b)* $\pm\frac{2\sqrt{2}}{3}$ *(c)* $\pm\frac{8}{9}$ *(d)* $\pm\frac{4\sqrt{2}}{3}$

60. The harmonic conjugate of (2, 3, 4) with respect to the points (3, −2, 2), (6, −17, −4) is

(a) $\left(2, \frac{1}{3}, \frac{1}{4}\right)$ *(b)* $\left(\frac{18}{5}, -5, \frac{4}{5}\right)$ *(c)* $\left(\frac{18}{5}, \frac{5}{4}, \frac{4}{5}\right)$ *(d)* $\left(\frac{18}{5}, -5, \frac{-4}{5}\right)$

61. If a line makes angles α, β, γ and δ with the four diagonals of a cube, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta$ is

(a) $\frac{4}{3}$ *(b)* $\frac{8}{3}$
(c) $\frac{7}{3}$ *(d)* $\frac{5}{3}$

62. If the plane $56x + 4y + 9z = 2016$ meets the coordinate axes in A, B, C, then the centroid of the ΔABC is

(a) (12, 168, 224) *(b)* (12, 168, 112)
(c) $\left(12, 168, \frac{224}{3}\right)$ *(d)* $\left(12, -168, \frac{224}{3}\right)$

63. The value(s) of x for which the function

$$f(x) = \begin{cases} 1 - x, & x < 1 \\ (1 - x)(2 - x), & 1 \le x \le 2 \\ 3 - x, & x > 2 \end{cases}$$

fails to be continuous is(are)

(a) 1 *(b)* 2 *(c)* 3 *(d)* all real numbers

64. $\lim\limits_{x \to 0} \frac{6^x - 3^x - 2^x + 1}{x^2}$ is equal to

(a) $(\log_e 2)\log_e 3$ *(b)* $\log_e 5$ *(c)* $\log_e 6$ *(d)* 0

65. Define $f(x) = \begin{cases} x^2 + bx + c, & x < 1 \\ x, & x \ge 1 \end{cases}$. If $f(x)$ is differentiable at $x = 1$, then $(b - c)$ is equal to

(a) −2 *(b)* 0 *(c)* 1 *(d)* 2

66. If $x = a$ is a root of multiplicity two of a polynomial equation $f(x) = 0$, then

(a) $f'(a) = f''(a) = 0$ *(b)* $f''(a) = f(a) = 0$
(c) $f'(a) \ne 0 \ne f''(a)$ *(d)* $f(a) = f'(a) = 0; f''(a) \ne 0$

67. If $y = \log_2(\log_2 x)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\log_e 2}{x\log_e x}$ *(b)* $\frac{1}{\log_e(2x)^x}$ *(c)* $\frac{1}{(x\log_e x)\log_e 2}$ *(d)* $\frac{1}{x(\log_2 x)^2}$

68. The angle of intersection between the curves $y^2 + x^2 = a^2\sqrt{2}$ and $x^2 - y^2 = a^2$ is

(a) $\frac{\pi}{3}$ *(b)* $\frac{\pi}{4}$ *(c)* $\frac{\pi}{6}$ *(d)* $\frac{\pi}{12}$

69. If $A > 0$, $B > 0$ and $A + B = \frac{\pi}{3}$, then the maximum value of $\tan A \tan B$ is

(a) $\frac{1}{\sqrt{3}}$ *(b)* $\frac{1}{3}$ *(c)* $\frac{1}{2}$ *(d)* $\sqrt{3}$

70. The equation of the common tangent drawn to the curves $y^2 = 8x$ and $xy = -1$ is

(a) $y = 2x + 1$ *(b)* $2y = x + 6$
(c) $y = x + 2$ *(d)* $3y = 8x + 2$

71. Suppose $f(x) = x(x + 3)(x - 2)$, $x \in [-1, 4]$. Then, a value of c in $(-1, 4)$ satisfying $f'(c) = 10$ is

(a) 2 *(b)* $\frac{5}{2}$
(c) 3 *(d)* $\frac{7}{2}$

72. If $\int x^3 e^{5x}\,dx = \frac{e^{5x}}{5^4}[f(x)] + C$, then $f(x)$ is equal to

(a) $\frac{x^3}{5} - \frac{3x^2}{5^2} + \frac{6x}{5^3} - \frac{6}{5^4}$ *(b)* $5x^3 - 5^2x^2 + 5^3x - 6$
(c) $5^2x^3 - 15x^2 + 30x - 6$ *(d)* $5^3x^3 - 75x^2 + 30x - 6$

73. $\int \frac{x}{(x^2+2x+2)^2}\,dx$ is equal to

(a) $\frac{x^2+2}{x^2+2x+2}-\frac{1}{2}\tan^{-1}(x+1)+C$

(b) $\frac{x^2+2}{2(x^2+2x+2)}-\frac{1}{2}\tan^{-1}(x-1)+C$

(c) $\frac{x^2-2}{4(x^2+2x+2)}-\frac{1}{2}\tan^{-1}(x+1)+C$

(d) $\frac{2(x-1)}{(x^2+2x+2)}+\frac{1}{2}\tan^{-1}(x+1)+C$

74. If $\int \log(a^2+x^2)dx = h(x)+C$, then $h(x)$ is equal to

(a) $x\log(a^2+x^2)+2\tan^{-1}\left(\frac{x}{a}\right)$

(b) $x^2\log(a^2+x^2)+x+a\tan^{-1}\left(\frac{x}{a}\right)$

(c) $x\log(a^2+x^2)-2x+2a\tan^{-1}\left(\frac{x}{a}\right)$

(d) $x^2\log(a^2+x^2)+2x-a^2\tan^{-1}\left(\frac{x}{a}\right)$

75. For $x>0$, if $\int (\log x)^5\,dx$ is equal to

$x[A(\log x)^5+B(\log x)^4+C(\log x)^3+D(\log x)^2+E(\log x)+F]$ + constant, then $A+B+C+D+E+F$ is equal to

(a) −44 (b) −42 (c) −40 (d) −36

76. The area included between the parabola $y=\frac{x^2}{4a}$ and the curve $y=\frac{8a^3}{(x^2+4a^2)}$ is

(a) $a^2\left(2\pi+\frac{2}{3}\right)$ (b) $a^2\left(2\pi-\frac{8}{3}\right)$

(c) $a^2\left(\pi+\frac{4}{3}\right)$ (d) $a^2\left(2\pi-\frac{4}{3}\right)$

77. By the definition of the definite integral, the value of $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2-1}}+\frac{1}{\sqrt{n^2-2^2}}+\ldots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right)$ is equal to

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

78. $\int_{-\pi/4}^{\pi/4}\frac{x+\frac{\pi}{4}}{2-\cos 2x}dx$ is equal to

(a) $\frac{8\pi\sqrt{3}}{5}$ (b) $\frac{2\pi\sqrt{3}}{9}$ (c) $\frac{4\pi^2\sqrt{3}}{9}$ (d) $\frac{\pi^2}{6\sqrt{3}}$

79. The solution of the differential equation $(1+y^2)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$, is

(a) $xe^{\tan^{-1}y}=\tan^{-1}y+C$ (b) $xe^{2\tan^{-1}y}=e^{-\tan^{-1}y}+C$

(c) $2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+C$ (d) $x^2e^{\tan^{-1}y}=4e^{2\tan^{-1}y}+C$

80. The solution of the differential equation $(2x-4y+3)\frac{dy}{dx}+(x-2y+1)=0$ is

(a) $\log[(2x-4y)+3]=x-2y+C$

(b) $\log[2(2x-4y)+3]=2(x-2y)+C$

(c) $\log[2(x-2y)+5]=2(x+y)+C$

(d) $\log[4(x-2y)+5]=4(x+2y)+C$

Answers with Explanation

1. *(d)* Given, $f(x)=\sqrt{\log_{0.5}x!}$

$f(x)$ is defined when

$\log_{0.5}x!\geq 0$

$\Rightarrow x!\leq(0.5)^0 \Rightarrow x!\leq 1$

$\therefore x\in\{0,1\}$

2. *(c)* Given, $f(x)=|x-1|+|x-2|+|x-3|$

$$\Rightarrow f(x)=\begin{cases}6-3x, & x<1\\ 4-x, & 1<x<2\\ x, & 2<x<3\\ 3x-6, & x>3\end{cases}$$

When, $2<x<3$, then $f(x)=x$

So, $f(x)=x$ is one-one and onto function.

$\therefore f(x)$ is a bijection.

3. *(b)* We have,

$(n+16)(n+17)(n+18)(n+19)$

These numbers are the product of four consecutive natural numbers.

∴ This number divides by $4!=24$.

4. *(a)* Let $A=\begin{vmatrix}a&b&c\\b&c&a\\c&a&b\end{vmatrix}$ [apply $C_1\to C_1+C_2+C_3$]

$$=\begin{vmatrix}a+b+c&b&c\\a+b+c&c&a\\a+b+c&a&b\end{vmatrix}=(a+b+c)\begin{vmatrix}1&b&c\\1&c&a\\1&a&b\end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix}1&b&c\\0&c-b&a-c\\0&a-b&b-c\end{vmatrix}$$

[apply $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$]

$=(a+b+c)[(c-b)(b-c)-(a-c)(a-b)]$

$=(a+b+c)[bc-b^2-c^2+bc-a^2+ab+ac-bc]$

$=-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$=-\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

Since, a, b, c are distinct positive numbers.

∴ The value of determinant A is less than 0.

5. ***(d)*** **Key Idea** If the area of a triangle formed by three points is zero, then the points are collinear.

We have x_1, x_2, x_3 and y_1, y_2, y_3 are in GP with the same common ratio.

Let r be the common ratio.

$\therefore \quad x_1 = x$

$x_2 = xr$ and $x_3 = xr^2$

Similarly, $y_1 = y$

$y_2 = yr$ and $y_3 = yr^2$

$$\therefore \quad \text{Area of } \Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x & y & 1 \\ xr & yr & 1 \\ xr^2 & yr^2 & 1 \end{vmatrix} = \frac{1}{2}xy\begin{vmatrix} 1 & 1 & 1 \\ r & r & 1 \\ r^2 & r^2 & 1 \end{vmatrix}$$

$= \frac{1}{2} \times 0 = 0$ [$\because C_1, C_2$ are identical]

$\therefore$ The given points are collinear.

6. ***(b)*** **Key Idea** If $D = D_x = D_y = D_z = 0$, then the system is consistent with infinitely many solution.

We have, $x - y + 2z = 4$

$3x + y + 4z = 6$

$x + y + z = 1$

Let $D = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(1-4) + 1(3-4) + 2(3-1)$

$= -3 - 1 + 4 = 0$

and $D_x = \begin{vmatrix} 4 & -1 & 2 \\ 6 & 1 & 4 \\ 1 & 1 & 1 \end{vmatrix}$

$= 4(1-4) + 1(6-4) + 2(6-1)$

$= -12 + 2 + 10 = 0$

Also, $D_y = D_z = 0,$

Now, $D = D_x = D_y = D_z = 0$

$\therefore$ These equations have infinitely many solutions.

7. ***(c)*** We have, $|z+3|^2 - |z-3|^2 = 15$

Let $z = x + iy$

$\therefore \quad |x + iy + 3|^2 - |x + iy - 3|^2 = 15$

$\Rightarrow \quad (x+3)^2 + y^2 - (x-3)^2 - y^2 = 15$

$\Rightarrow \quad x^2 + 6x + 9 - x^2 + 6x - 9 = 15$

$\Rightarrow \quad 12x = 15$

$\Rightarrow \quad 4x = 5$

which represents a straight line.

8. ***(a)*** We have,

$$\frac{(1+i)^{2016}}{(1-i)^{2014}} = \frac{[(1+i)^2]^{1008}}{[(1-i)^2]^{1007}} = \frac{(1+2i+i^2)^{1008}}{(1-2i+i^2)^{1007}}$$

$$= \frac{(2i)^{1008}}{(-2i)^{1007}} \quad [\because i^2 = -1]$$

$= -(2i)^{1008 - 1007}$

$= -2i$

9. ***(d)*** Given, $|z_1| = 1, |z_2| = 2, |z_3| = 3$

and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

Now, $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

$\Rightarrow \quad \begin{vmatrix} |z_3|^2 z_1z_2 + |z_2|^2 z_1z_3 \\ + |z_1|^2 z_2z_3 \end{vmatrix} = 12$

$\Rightarrow \quad \begin{vmatrix} z_3\bar{z}_3z_1z_2 + z_2\bar{z}_2z_1z_3 \\ + z_1\bar{z}_1z_2z_3 \end{vmatrix} = 12$ [$\because |z|^2 = z\bar{z}$]

$\Rightarrow \quad |z_1z_2z_3(\bar{z}_3 + \bar{z}_2 + \bar{z}_1)| = 12$

$\Rightarrow \quad |z_1z_2z_3||\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |z_1||z_2||z_3||\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad 1 \times 2 \times 3|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 2$

$\therefore \quad |z_1 + z_2 + z_3| = 2$ [$\because |\bar{z}| = |z|$]

10. ***(c)*** Given, $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the nth roots of unity.

$\therefore \; z^n - 1 = (z-1)(z-z_1)(z-z_2), \ldots, (z-z_{n-1})$

$\Rightarrow \quad (z-1)(z^{n-1} + z^{n-2} + \ldots + z^2 + z + 1)$

$= (z-1)(z-z_1)\ldots(z-z_{n-1})$

$\Rightarrow \quad (z^{n-1} + z^{n-2} + \ldots + z^2 + z + 1) = (z-z_1)$

$(z-z_2)\ldots(z-z_{n-1})$

Put $z = 1$, we get $(1 + 1 + \ldots n \text{ times})$

$= (1-z_1)(1-z_2) + \ldots + (1-z_{n-1})$

$\therefore \quad n = (1-z_1)(1-z_2)\ldots(1-z_{n-1})$

11. ***(b)*** Given, $(12)^{4+2x^2} = (24\sqrt{3})^{3x^2-2}$

$\Rightarrow \quad (12)^{4+2x^2} = (12)^{\frac{3}{2}(3x^2-2)}$

On comparing powers both sides, we get

$\Rightarrow \quad 4 + 2x^2 = \frac{3}{2}(3x^2 - 2)$

$\Rightarrow \quad 8 + 4x^2 = 9x^2 - 6$

$\Rightarrow \quad 5x^2 = 14 \Rightarrow x^2 = \frac{14}{5}$

$\therefore \quad x = \pm\sqrt{\frac{14}{5}}$

12. ***(a)*** Given, $|x|^2 - 5|x| - 24 = 0$

$\Rightarrow \quad |x|^2 - 8|x| + 3|x| - 24 = 0$

$\Rightarrow \quad (|x| - 8)(|x| + 3) = 0$

$\Rightarrow \quad |x| = 8$ or $|x| = -3$ is rejected.

$\Rightarrow \quad |x| = 8$

$\therefore \quad x = \pm 8$

Hence, product of roots $= 8 \times (-8) = -64$

and sum of roots $= +8 - 8 = 0$

13. ***(a)*** Let $f(x) = x^5 + 3x^3 + 4x + 30$

$\Rightarrow \quad f'(x) = 5x^4 + 9x^2 + 4$

$\Rightarrow \quad f'(x) > 0, \forall\, x \in R$

Hence, the $f(x) = 0$ has only one real root.

14. ***(b)*** **Key Idea** An equation whose roots are k times of the roots of a given equation is obtained by replacing x by x/k in the given equation.

We have,

$$x^3 + \frac{1}{4}x^2 - \frac{1}{16}x + \frac{1}{144} = 0$$

The equation whose roots are k times the roots of the given equation, is

$$\left(\frac{x}{k}\right)^3+\frac{1}{4}\left(\frac{x}{k}\right)^2-\frac{1}{16}\left(\frac{x}{k}\right)+\frac{1}{144}=0$$

$\Rightarrow \quad x^3+\frac{k}{4}x^2-\frac{k^2}{16}x+\frac{k^3}{144}=0$

Since, the coefficients of the equation are integer.

$\therefore \quad k=12$

15. ***(c)*** We have, digits 2, 3, 4, 5, 6

Sum of all 4-digit numbers formed by the using 2, 3, 4, 5, 6 without repetition

$$={}^4P_3\,(2+3+4+5+6)(10^3+10^2+10^1+10^0)$$

$$=24(20)\,(1000+100+10+1)=(480)\,(1111)=533280$$

16. ***(c)*** The required number of ways of selecting two subsets will be $3^5=243$.

17. ***(a)*** We have, $(1-x+x^2-x^3)^4$

$$=(1+x^2-x-x^3)^4$$

$$=[(1+x^2)-x(1+x^2)]^4$$

$$=[(1+x^2)\,(1-x)]^4=(1+x^2)^4\,(1-x)^4$$

$$=(1+4x^2+6x^4+4x^6+x^8)$$

$$(1-4x+6x^2-4x^3+x^4)$$

$\therefore$ Coefficient of $x^4=1+4\times6+6=1+24+6=31$

18. ***(a)*** Middle term of the expansion $(1+x)^{2n}$ is ${}^{2n}C_n x^n$.

Since, middle term is the greatest term.

$\therefore \quad {}^{2n}C_n x^n > {}^{2n}C_{n-1}x^{n-1}$

and $\quad {}^{2n}C_n x^n > {}^{2n}C_{n+1}x^{n+1}$

$\Rightarrow \quad x>\frac{{}^{2n}C_{n-1}}{{}^{2n}C_n}$ and $x<\frac{{}^{2n}C_n}{{}^{2n}C_{n+1}}$

Hence, $\quad x\in\left(\frac{{}^{2n}C_{n-1}}{{}^{2n}C_n},\frac{{}^{2n}C_n}{{}^{2n}C_{n+1}}\right)$

$\therefore \quad x\in\left(\frac{n}{n+1},\frac{n+1}{n}\right)$

19. ***(c)*** We have,

$$\frac{3x}{(x-2)(x-1)}=\frac{6}{x-2}-\frac{3}{x-1}$$

$$=6(x-2)^{-1}-3(x-1)^{-1}$$

$$=-3\left(1-\frac{x}{2}\right)^{-1}+3(1-x)^{-1}$$

It is valid iff $\left|\frac{x}{2}\right|<1$ and $|x|<1$.

$\Rightarrow \quad |x|<2$ and $|x|<1$

$\Rightarrow \quad x\in(-2,2)$ and $x\in(-1,1)$

$\therefore \quad x\in(-1,1)$

Hence, $-1<x<1$

20. ***(a)*** We have,

$$(1+\tan\alpha)\,(1+\tan4\alpha)=2$$

$\Rightarrow \quad 1+\tan\alpha+\tan4\alpha+\tan\alpha\tan4\alpha=2$

$\Rightarrow \quad \tan\alpha+\tan4\alpha=1-\tan\alpha\tan4\alpha$

$\Rightarrow \quad \frac{\tan\alpha+\tan4\alpha}{1-\tan\alpha\tan4\alpha}=1$

$\Rightarrow \quad \tan(\alpha+4\alpha)=1$

$\Rightarrow \quad \tan(\alpha+4\alpha)=\tan\frac{\pi}{4}$

$\Rightarrow \quad \alpha+4\alpha=\frac{\pi}{4} \quad \Rightarrow \quad 5\alpha=\frac{\pi}{4}=\frac{\pi}{20}$

21. ***(a)*** Given, $\cos\theta=\frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$

On applying componendo and dividendo, we get

$$\frac{\cos\theta+1}{\cos\theta-1}=\frac{\cos\alpha-\cos\beta+1-\cos\alpha\cos\beta}{\cos\alpha-\cos\beta-1+\cos\alpha\cos\beta}$$

$\Rightarrow \quad \frac{\cos\theta+1}{\cos\theta-1}=\frac{(\cos\alpha+1)\,(1-\cos\beta)}{(\cos\alpha-1)\,(\cos\beta+1)}$

$\Rightarrow \quad \frac{1+\cos\theta}{1-\cos\theta}=\frac{(1+\cos\alpha)\,(1-\cos\beta)}{(1-\cos\alpha)\,(1+\cos\beta)}$

$\Rightarrow \quad \frac{2\cos^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=\frac{2\cos^2\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}}\times\frac{2\sin^2\frac{\beta}{2}}{2\cos^2\frac{\beta}{2}}$

$\Rightarrow \quad \cot^2\frac{\theta}{2}=\cot^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}$

$\therefore \quad \tan\frac{\theta}{2}=\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$

22. ***(d)*** We have,

$$\frac{1+\sin2\alpha}{\cos(2\alpha-2\pi)\tan\left(\alpha-\frac{3\pi}{4}\right)}$$

$$-\frac{1}{4}\sin2\alpha\left[\cot\frac{\alpha}{2}+\cot\left(\frac{3\pi}{2}+\frac{\alpha}{2}\right)\right]$$

$\Rightarrow$

$$\frac{(\cos\alpha+\sin\alpha)^2}{\cos2\alpha\left(\frac{\tan\alpha-\tan\frac{3\pi}{4}}{1+\tan\alpha\tan\frac{3\pi}{4}}\right)}$$

$$-\frac{1}{4}2\sin\alpha\cos\alpha\left(\cot\frac{\alpha}{2}-\tan\frac{\alpha}{2}\right)$$

$$=\frac{(\cos\alpha+\sin\alpha)^2}{(\cos^2\alpha-\sin^2\alpha)\left(\frac{\tan\alpha+1}{1-\tan\alpha}\right)}$$

$$-\frac{1}{4}\times2\sin\alpha\cos\alpha\left(\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}-\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}\right)$$

$$=\frac{(\cos\alpha+\sin\alpha)}{(\cos\alpha-\sin\alpha)\left(\frac{\sin\alpha+\cos\alpha}{\cos\alpha-\sin\alpha}\right)}$$

$$-\frac{1}{2}\sin\alpha\cos\alpha\left(\frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)$$

$$=1-\frac{\sin\alpha\cos^2\alpha}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$=1-\frac{\sin\alpha\cos^2\alpha}{\sin\alpha}$$

$$=1-\cos^2\alpha=\sin^2\alpha$$

23. ***(b)*** **Key Idea** If $\cos x = \cos y$

$\Rightarrow \quad x = 2n\pi \pm y$

Given, $\frac{1}{6}\sin\theta, \cos\theta, \tan\theta$ are in GP.

$\therefore \quad \cos^2\theta = \frac{1}{6}\sin\theta \times \tan\theta$

$\Rightarrow \quad \cos^2\theta = \frac{\sin^2\theta}{6\cos\theta}$

$\Rightarrow \quad 6\cos^3\theta - \sin^2\theta = 0$

$\Rightarrow \quad 6\cos^3\theta - 1 + \cos^2\theta = 0$

$\Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0$

$\Rightarrow (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

For $3\cos^2\theta + 2\cos\theta + 1 = 0$

value of $\cos\theta$ is imaginary.

Hence, consider only.

$\Rightarrow \quad 2\cos\theta - 1 = 0 \Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{3}$

$\therefore \quad \theta = 2n\pi \pm \frac{\pi}{3}$

24. ***(a)*** Given, $x = \sin(2\tan^{-1}2)$

and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$

$\Rightarrow \quad x = \sin(2\tan^{-1}2)$

Let $\tan^{-1}2 = \theta \Rightarrow \tan\theta = 2$

$\therefore \quad x = \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2(2)}{1+(2)^2} = \frac{4}{5}$

Now, $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$

Let $\tan^{-1}\frac{4}{3} = \phi$

$\tan\phi = \frac{4}{3}, \cos\phi = \frac{3}{5}$

Then, $y = \sin\left(\frac{\phi}{2}\right)$

$= \sqrt{\frac{1-\cos\phi}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{1}{5}}$

$\therefore \quad x > y$

25. ***(c)*** $\cosh(x) = \frac{5}{4}$

$\cosh(3x) = 4\cosh^3(x) - 3\cosh(x)$

$= 4\left(\frac{5}{4}\right)^3 - 3\times\frac{5}{4} = \frac{125}{16} - \frac{15}{4}$

$= \frac{125-60}{16} = \frac{65}{16}$

26. ***(b)*** Given, $x = \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}$

$y = \tan\left(\frac{C-A}{2}\right)\tan\frac{B}{2}$

and $z = \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2}$

$\Rightarrow \quad x = \frac{b-c}{b+c} \qquad \left[\because \tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}\right]$

$y = \frac{c-a}{c+a}$ and $z = \frac{a-b}{a+b}$

Now, $\frac{1+x}{1-x} = \frac{b+c+b-c}{b+c-b+c}$

[by componendo and dividendo]

$\Rightarrow \quad \frac{1+x}{1-x} = \frac{b}{c}$

Similarly, $\frac{1+y}{1-y} = \frac{c}{a}$ and $\frac{1+z}{1-z} = \frac{a}{b}$

$\therefore \left(\frac{1+x}{1-x}\right)\left(\frac{1+y}{1-y}\right)\left(\frac{1+z}{1-z}\right) = \frac{b}{c}\times\frac{c}{a}\times\frac{a}{b} = 1$

$\Rightarrow \quad (1+x)(1+y)(1+z) = (1-x)(1-y)(1-z)$

$\Rightarrow \quad 1 + x + y + z + xy + yz + zx + xyz$

$= 1 - (x + y + z) + xy + yz + zx - xyz$

$\Rightarrow \quad x + y + z = -xyz$

27. ***(d)*** Given, a, b, c are in GP.

$\therefore \quad b^2 = ac$ ...(i)

and $C - A = 60°$ ...(ii)

$\Rightarrow \quad \cos(C - A) = \cos 60°$

$\Rightarrow \quad \cos C\cos A + \sin C\sin A = \frac{1}{2}$

$\Rightarrow 2\cos C\cos A + 2\sin C\sin A = 1$

$\Rightarrow \cos(C + A) + \cos(C - A) + 2k^2ac = 1$

$\left[\because \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k\right]$

$\Rightarrow \quad \cos(\pi - B) + \cos 60° + 2k^2b^2 = 1 \qquad [\because ac = b^2]$

$\Rightarrow \quad -\cos B + \frac{1}{2} + 2\sin^2 B = 1 \qquad [\because bk = \sin B]$

$\Rightarrow \quad -2\cos B + 1 + 4\sin^2 B = 2$

$\Rightarrow \quad -2\cos B + 1 + 4(1 - \cos^2 B) = 2$

$\Rightarrow \quad 4\cos^2 B + 2\cos B - 3 = 0$

$\therefore \quad \cos B = \frac{-2 \pm \sqrt{4+48}}{2\times 4} = \frac{-2 \pm 2\sqrt{13}}{8}$

$= \frac{\sqrt{13}-1}{4}$ or $\frac{-\sqrt{13}-1}{4}$

28. ***(c)*** We know that,

$r_2 + r_3 = 4R\cos^2\frac{A}{2}$

$\Rightarrow \quad r_2 + r_3 = 4R\cos^2 45° \qquad [\because \angle A = 90°]$

$\Rightarrow \quad r_2 + r_3 = 4R\times\frac{1}{2}$

$\Rightarrow \quad r_2 + r_3 = 2R$

$\therefore \cos^{-1}\left(\frac{R}{r_2 + r_3}\right) = \cos^{-1}\left(\frac{R}{2R}\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60°$

29. ***(d)*** Given,

$\gamma = (1 + \lambda - \mu)\hat{i} + (2 - \lambda)\hat{j} + (3 - 2\lambda + 2\mu)\hat{k}$

$= (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} - 2\hat{k}) + \mu(-\hat{i} + 2\hat{k})$

Equation of plane is given by

$[\mathbf{r} - (\hat{i} + 2\hat{j} + 3\hat{k})]\times[(\hat{i} - \hat{j} - 2\hat{k})\times(-\hat{i} + 2\hat{k})] = 0$

$\Rightarrow \quad [\mathbf{r} - (\hat{i} + 2\hat{j} + 3\hat{k})]\times(-2\hat{i} - \hat{k}) = 0$

$\Rightarrow \quad \mathbf{r} \times (2\hat{\mathbf{i}} + \hat{\mathbf{k}}) - 5 = 0$

$\Rightarrow \quad \mathbf{r} \times (2\hat{\mathbf{i}} + \hat{\mathbf{k}}) = 5$

Now, equation of plane in cartesian form is

$(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \times (2\hat{\mathbf{i}} + \hat{\mathbf{k}}) = 5$

$\therefore \quad 2x + z = 5$

30. ***(b)*** We have, $\mathbf{r} = 3\mathbf{p} + 4\mathbf{q}$ ...(i)

and $2\mathbf{r} = \mathbf{p} - 3\mathbf{q}$ ...(ii)

On eliminating $\mathbf{p}$, from Eqs. (i) and (ii), we get

$\mathbf{r} = 3(2\mathbf{r} + 3\mathbf{q}) + 4\mathbf{q}$

$\mathbf{r} = 6\mathbf{r} + 9\mathbf{q} + 4\mathbf{q}$

$\Rightarrow \quad 5\mathbf{r} = -13\mathbf{q}$

$\Rightarrow \quad \mathbf{r} = -\frac{13}{5}\mathbf{q}$

$\therefore$ $\mathbf{r}$ and $\mathbf{q}$ have opposite directions.

Hence, $|\mathbf{r}| > 2|\mathbf{q}|$

31. ***(a)*** Given, $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

$\mathbf{b} = m\hat{\mathbf{i}} + n\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$

and $\mathbf{a} \times \mathbf{b} = 0$

Since, $\mathbf{a} \times \mathbf{b} = 0$

$\therefore \quad \mathbf{a} \parallel \mathbf{b}$

Hence, $\frac{2}{m} = \frac{3}{n} = \frac{-5}{12}$

$\Rightarrow \quad m = -\frac{24}{5}$

and $n = -\frac{36}{5}$

32. ***(d)*** Given, $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and angle between $\mathbf{a}$ and $\mathbf{b} = 120°$.

$$\therefore \quad |4\mathbf{a} + 3\mathbf{b}|^2 = (4\mathbf{a} + 3\mathbf{b}) \times (4\mathbf{a} + 3\mathbf{b})$$
$$= 16|\mathbf{a}|^2 + 9|\mathbf{b}|^2 + 24\,\mathbf{a} \times \mathbf{b}$$
$$= 16|\mathbf{a}|^2 + 9|\mathbf{b}|^2 + 24|\mathbf{a}||\mathbf{b}|\cos\theta$$
$$= 16(3)^2 + 9(4)^2 + 24 \times 3 \times 4 \times \cos 120°$$
$$= 16 \times 9 + 9 \times 16 + 24 \times 12 \times \left(-\frac{1}{2}\right)$$
$$= 144 + 144 - 144 = 144$$

So, $|4\mathbf{a} + 3\mathbf{b}| = \sqrt{144} = 12$

33. ***(a)*** Given, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$

$(\mathbf{c} \times \mathbf{a})\mathbf{b} - (\mathbf{c} \times \mathbf{b})\mathbf{a} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$

$\Rightarrow \quad 0 - (|\mathbf{c}||\mathbf{b}|\cos\theta)\mathbf{a} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$ $[\because \mathbf{c} \times \mathbf{a} = 0 \Rightarrow \mathbf{c} \perp \mathbf{a}]$

$\Rightarrow \quad \cos\theta = -\frac{1}{3}$

$\Rightarrow \quad \cos^2\theta = \frac{1}{9}$

$\Rightarrow \quad 1 - \sin^2\theta = \frac{1}{9}$

$\Rightarrow \quad \sin^2\theta = 1 - \frac{1}{9} = \frac{8}{9}$

$\therefore \quad \sin\theta = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

34. ***(c)*** **Key Idea** For three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ if $[\mathbf{x}\ \mathbf{y}\ \mathbf{z}] = 0$ then vectors are coplanar.

We have,

$$a(\vec{\alpha} \times \vec{\beta}) + b(\vec{\beta} \times \vec{\gamma}) + c(\vec{\gamma} \times \vec{\alpha}) = 0$$

On multiplying by α, we get

$$a\,\vec{\alpha} \times (\vec{\alpha} \times \vec{\beta}) + b\,\vec{\alpha} \times (\vec{\beta} \times \vec{\gamma}) + c\,\vec{\alpha} \times (\vec{\gamma} \times \vec{\alpha}) = 0$$
$$a[\vec{\alpha}\,\vec{\alpha}\,\vec{\beta}] + b[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] + c[\vec{\alpha}\,\vec{\gamma}\,\vec{\alpha}] = 0$$

$\Rightarrow \quad b[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = 0$

Since, $b \neq 0$

$\therefore \quad [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = 0$

Hence, $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are coplanar.

35. ***(c)*** Given, $\bar{x} = 50$

and $\sum_{i=1}^{10}(x_i - \bar{x})^2 = 250$

We know that,

$$\sigma^2 \text{ (variance)} = \sum_{i=1}^{n}\frac{|x_1 - \bar{x}|^2}{n}$$

$$\Rightarrow \quad \sigma^2 = \sum_{i=1}^{10}\frac{|x_1 - \bar{x}|^2}{10} = \frac{250}{10} = 25$$

$\Rightarrow \quad \sigma = 5$

$\therefore$ Coefficient of variation (CV) $= \frac{\sigma}{\bar{x}} \times 100 = \frac{5}{50} \times 100 = 10$

36. ***(b)*** Mean of first 50 even natural numbers

i.e. $\bar{x} = \frac{2 + 4 + 6 + ... + 98 + 100}{50}$

$$\Rightarrow \quad \bar{x} = \frac{2(1 + 2 + 3 + ... + 49 + 50)}{50}$$

$$\Rightarrow \quad \bar{x} = \frac{2 \times 50 \times 51}{2 \times 50} = 51$$

Variance of these numbers

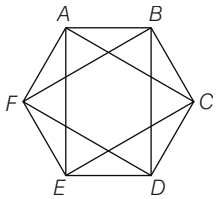
$$\therefore \quad \sigma^2 = \frac{\Sigma x_i^2}{n} - (\bar{x})^2$$
$$= \frac{(2^2 + 4^2 + 6^2 + ... + 98^2 + 100^2)}{50} - (51)^2$$
$$= \frac{2^2(1^2 + 2^2 + 3^2 + ... + 49^2 + 50^2) - (51)^2 \times 50}{50}$$
$$= \frac{4(50)(51)(101) - (51) \times (51) \times 50 \times 6}{6 \times 50}$$
$$= \frac{50 \times 51(404 - 306)}{6 \times 50}$$
$$= \frac{51 \times 98}{6} = 833$$

37. ***(c)*** Total number of outcomes $= {}^6C_3$

$$= \frac{6!}{3!3!} = 20$$

Only two equilateral triangles are possible of regular hexagon, i.e. ΔAEC and ΔBFD.

$\therefore$ Required probability $= \frac{2}{20} = \frac{1}{10}$



38. *(a)* Given, $P(A)=75\%=\frac{75}{100}=\frac{3}{4}$

and $P(B)=80\%=\frac{80}{100}=\frac{4}{5}$

$\therefore$ Required probability $=P(A\bar{B})+P(\bar{A}B)$

$=P(A)\times P(\bar{B})+P(\bar{A})\times P(B)$

$=P(A)\times[1-P(B)]+[1-P(A)]\times P(B)$

$=\frac{3}{4}\times\left(1-\frac{4}{5}\right)+\left(1-\frac{3}{4}\right)\times\frac{4}{5}$

$=\frac{3}{4}\times\frac{1}{5}+\frac{1}{4}\times\frac{4}{5}=\frac{3+4}{20}=\frac{7}{20}$

39. *(d)* Given, mean and variance of a binomial distribution are 4 and 2, respectively.

$\therefore \quad \mu=np=4 \quad \ldots(i)$

and $\sigma^2=npq=2 \quad \ldots(ii)$

On solving Eqs. (i) and (ii), we get

$n=8,\ p=\frac{1}{2}, q=\frac{1}{2}$

$\therefore$ Required probability

$P(X=2)={}^8C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^6 \quad [\because P[X=r]={}^nC_r p^r q^{n-r}]$

$=\frac{{}^8C_2}{2^8}=\frac{28}{256}=\frac{7}{64}$

40. *(a)* We know that in Poisson distribution,

$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$

$\lambda=$ mean of distribution $=np$

$k=$ probability of success

Here, $p=\frac{10}{50}=\frac{1}{5}$ and $n=1$

$\therefore \quad \lambda=1\times\frac{1}{5}=0.2$

$\therefore$ Required probability

$P(X\geq 3)=P(X=3)+P(X=4)+\ldots=\sum_{k=3}^{\infty}\frac{e^{-\lambda}\lambda^k}{k!},\ \lambda=0.2$

41. *(a)* **Key Idea** If vertices of a triangle are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the coordinate of centroid will be $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

We have,

$A=(a\cos k, a\sin k)$

$B=(b\sin k, -b\cos k)$

$C=(1, 0)$

Let (x, y) be the centroid of ΔABC.

$\therefore \quad x=\frac{a\cos k+b\sin k+1}{3}$

and $y=\frac{a\sin k-b\cos k+0}{3}$

$\Rightarrow 3x-1=a\cos k+b\sin k \quad \ldots(i)$

and $3y=a\sin k-b\cos k \quad \ldots(ii)$

On squaring and then adding Eqs. (i) and (ii), we get

$(3x-1)^2+(3y)^2=a^2(\sin^2 k+\cos^2 k)+b^2(\sin^2 k+\cos^2 k)$

$\therefore \quad (3x-1)^2+9y^2=a^2+b^2$

$\therefore \quad (1-3x)^2+9y^2=a^2+b^2$

42. *(c)* Coordinate axes are rotated through an angle of $\frac{\pi}{6}$ about the origin.

Now, axis is $x\cos\frac{\pi}{6}-y\sin\frac{\pi}{6}$ and $x\sin\frac{\pi}{6}+y\cos\frac{\pi}{6}$.

Substitute these values in $\sqrt{3}x^2-4xy+\sqrt{3}y^2=0$, we get

$\sqrt{3}\left(x\cos\frac{\pi}{6}-y\sin\frac{\pi}{6}\right)^2-4\left(x\cos\frac{\pi}{6}-y\sin\frac{\pi}{6}\right)\left(x\sin\frac{\pi}{6}+y\cos\frac{\pi}{6}\right)+\sqrt{3}\left(x\sin\frac{\pi}{6}+y\cos\frac{\pi}{6}\right)^2=0$

$\Rightarrow \sqrt{3}\left(\frac{\sqrt{3}x}{2}-\frac{1}{2}y\right)^2-4\left(\frac{\sqrt{3}x-y}{2}\right)\left(\frac{x+\sqrt{3}y}{2}\right)+\sqrt{3}\left(\frac{x+\sqrt{3}y}{2}\right)^2=0$

$\Rightarrow \sqrt{3}(3x^2-2\sqrt{3}xy+y^2)-4(\sqrt{3}x^2+3xy-xy-\sqrt{3}y^2)+\sqrt{3}(x^2+2\sqrt{3}xy+3y^2)=0$

[on multiplying both sides by 4]

$\Rightarrow 3\sqrt{3}x^2-6xy+\sqrt{3}y^2-4\sqrt{3}x^2-12xy+4xy+4\sqrt{3}y^2+\sqrt{3}x^2+6xy+3\sqrt{3}y^2=0$

$\Rightarrow \quad 8\sqrt{3}y^2-8xy=0$

$\therefore \quad \sqrt{3}y^2-xy=0$

43. *(d)* **Key Idea** Two or more lines are concurrent, if they passes through a common point.

Since, $x+3y-9=0, 4x+by-2=0$

and $2x-y-4=0$ are concurrent.

$\therefore \quad \begin{vmatrix}1 & 3 & -9\\ 4 & b & -2\\ 2 & -1 & -4\end{vmatrix}=0$

$\Rightarrow 1(-4b-2)-3(-16+4)-9(-4-2b)=0$

$\Rightarrow -4b-2+36+36+18b=0$

$\Rightarrow 14b=-70$

$\Rightarrow b=-5$

On solving equations $x+3y-9=0$

and $2x-y-4=0$,

we get $x=3$ and $y=2$

$\therefore$ Concurrency point of line is (3, 2).

Equation of line passing through $(-5, 0)$ and $(3, 2)$ is

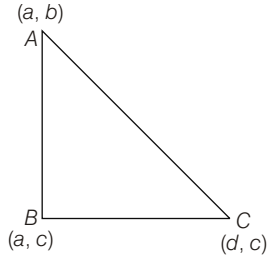
$y-0=\frac{2-0}{3+5}(x+5)$

$\Rightarrow \quad y=\frac{1}{4}(x+5)$

$\Rightarrow \quad 4y=x+5$

$\therefore \quad x-4y+5=0$

44. *(a)* **Key Idea** Orthocentre of a right angled triangle is that vertex which contains right angle.



Centroid of ΔABC

$=\left(\frac{a+a+d}{3}, \frac{b+c+c}{3}\right)$

$\equiv\left(\frac{2a+d}{3}, \frac{2c+b}{3}\right)$

Since, ΔABC is right triangle and $\angle B = 90°$

$\therefore$ Orthocentre of ΔABC is (a, c).

Mid-point of centroid and orthocentre is

$$= \left(\frac{\frac{2a+d}{3}+a}{2}, \frac{\frac{2c+b}{3}+c}{2}\right) = \left(\frac{5a+d}{6}, \frac{5c+b}{6}\right)$$

45. ***(c)*** Image of (1, 1) with respect to line $x + y + 5 = 0$ is

$$\frac{x-1}{1} = \frac{y-1}{1} = -\frac{2(1+1+5)}{1+1}$$

$\Rightarrow \quad x - 1 = y - 1 = -7$

$\Rightarrow \quad x = -6, y = -6$

$\therefore$ Image $(-6, -6)$

Now, distance from origin

$$= \sqrt{(0+6)^2 + (0+6)^2} = \sqrt{72} = 6\sqrt{2}$$

46. ***(d)*** Given, $x^2 + y^2 = 9$ and $x + y = 3$

Equation of pair of lines joining the origin to the points of intersection of the given line is

$$x^2 + y^2 = 9\left(\frac{x+y}{3}\right)^2 \quad \left[\because x + y = 3 \Rightarrow \frac{x+y}{3} = 1\right]$$

$\Rightarrow \quad x^2 + y^2 = (x + y)^2$

$\Rightarrow \quad x^2 + y^2 = x^2 + y^2 + 2xy$

$\therefore \quad xy = 0$

47. ***(a)*** **Key Idea** Orthocentre is the intersecting point of the altitude of the triangle.

We have, $x + y = 1$

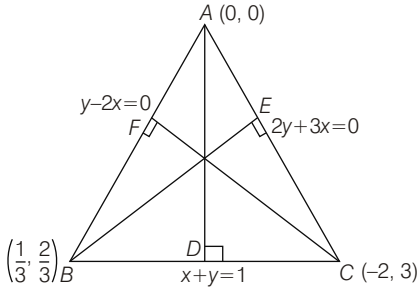
and $2y^2 - xy - 6x^2 = 0$

$\Rightarrow \quad 2y^2 - 4xy + 3xy - 6x^2 = 0$

$\Rightarrow \quad (2y + 3x)(y - 2x) = 0$

$\therefore$ Equation of sides of ΔABC are

$x + y = 1, 2y + 3x = 0$ and $y - 2x = 0$



On solving these equations simultaneously, we get $A(0, 0)$, $B\left(\frac{1}{3}, \frac{2}{3}\right)$ and $C(-2, 3)$.

Equation of altitude AD is

$$x - y = 0 \qquad \ldots(i)$$

and equation of altitude CF is

$$x + 2y = \lambda$$

Since, this passes through $(-2, 3)$.

$\therefore \quad -2 + 6 = \lambda \Rightarrow \lambda = 4$

Then, equation of altitude

$$CF \equiv x + 2y = 4 \qquad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x = \frac{4}{3}, y = \frac{4}{3}$$

$\therefore$ Orthocentre of ΔABC is $\left(\frac{4}{3}, \frac{4}{3}\right)$

48. ***(b)*** We have,

$$2x^2 - 3xy - 2y^2 + 10x + 5y = 0$$

$\Rightarrow \quad (2x + y)(x - 2y + 5) = 0$

Now, equation of line passing through origin and intersecting point of above lines is $2x + y = 0$.

Slope of this line is -2.

Since, this line is perpendicular to the line $kx + y + 3 = 0$.

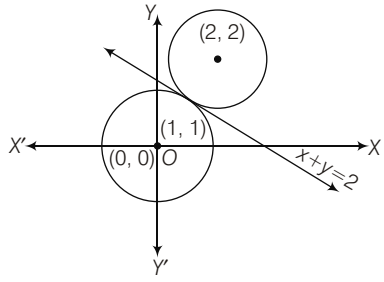
$\therefore \quad (-2)(-k) = -1 \Rightarrow k = -\frac{1}{2}$

49. ***(c)*** Equation of line at (1, 1) is $x + y - 2 = 0$.

Slope of line is -1.

Slope of line perpendicular to this line is 1.

$\therefore \quad \theta = \frac{\pi}{4}$



Centre of circle (h, k)

i.e. $x = h \pm r\cos\theta$ and $y = k \pm r\sin\theta$

Since, circle passes through (1, 1).

$\therefore \quad h = 1 \pm \sqrt{2}\cos\frac{\pi}{4}$

$$k = 1 \pm \sqrt{2}\sin\frac{\pi}{4}$$

$\Rightarrow \quad h = 1 \pm \frac{\sqrt{2}}{\sqrt{2}}$ and $k = 1 \pm \frac{\sqrt{2}}{\sqrt{2}}$

$\Rightarrow \quad h = 2, 0$ and $k = 2, 0$

$\therefore$ Centre $\equiv (2, 2)$ or $(0, 0)$

Hence, equation of circle is

$x^2 + y^2 = 2$ or $(x-2)^2 + (y-2)^2 = 2$

$\therefore$ Length of tangent from $(1, 2) = \sqrt{1^2 + 2^2 - 2} = \sqrt{3}$

50. ***(a)*** Equation of circle is

$$x^2 + y^2 - 2x - 2y - 3 = 0$$

Centre of circle $= (1, 1)$

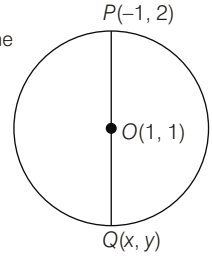
Since, PQ is normal to circle.

$\therefore$ PQ is diameter of circle and O is the mid-point of PQ.

$\therefore \quad \frac{x-1}{2} = 1, \frac{y+2}{2} = 1$

$\Rightarrow \quad x = 3, y = 0$

$\therefore$ Coordinates of $Q \equiv (3, 0)$.



51. ***(b)*** The lines $kx+2y-4=0$ and $5x-2y-4=0$ are conjugate with respect to the circle

$$x^2+y^2-2x-2y+1=0$$

We know that, if the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are conjugate with respect to the circle

$x^2+y^2+2gx+2fy+c=0$, then

$$r^2(a_1a_2+b_1b_2)=(a_1g+b_1f-c_1)(a_2g+b_2f-c_2)$$

From the above equation,

$$a_1=k, b_1=2, c_1=-4$$
$$a_2=5, b_2=-2, c_2=-4$$
$$g=-1, f=-1, c=1,$$
$$r^2=g^2+f^2-c$$
$$=1+1-1=1$$

Now,

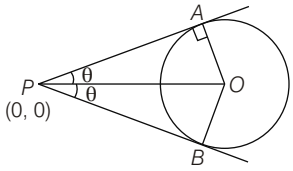
$$1(5k-4)=(-k-2+4)(-5+2+4)$$

$\Rightarrow \quad 5k-4=-k+2$

$\Rightarrow \quad 6k=6 \Rightarrow k=1$

52. ***(c)*** Given,

$$x^2+y^2+4x-6y+4=0$$



Radius of circle $=\sqrt{(2)^2+(-3)^2-4}$ $[\because \text{Radius}=\sqrt{g^2+f^2-c}]$

$$=\sqrt{4+9-4}=3$$

Length of tangent

$$PA=PB$$
$$=\sqrt{(0)^2+(0)^2+4(0)-6(0)+4}$$

$[\because \text{Length of tangent}=\sqrt{S_1}]$

$$=\sqrt{4}=2$$

In ΔOAP, $\tan\theta=\frac{OA}{PA}=\frac{3}{2}$

Now, $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2\times\frac{3}{2}}{1-\left(\frac{3}{2}\right)^2}=\frac{-12}{5}$

$\Rightarrow \quad 2\theta=\tan^{-1}\left(\frac{-12}{5}\right)$

Hence, the angle between the tangent drawn from the origin is $\tan^{-1}\left(\frac{-12}{5}\right)$

53. ***(d)*** Given, $x^2+y^2-2x-4y+c=0$ …(i)

and $x^2+y^2-4x-2y+4=0$ …(ii)

Centre of first circle is (1, 2) and radius of this circle is $\sqrt{1+4-c}=\sqrt{5-c}$

Centre of second circle is (2, 1) and radius of this circle is $\sqrt{4+1-4}=\sqrt{1}=1$

Hence, $C_1\equiv(1,2), r_1=\sqrt{5-c}$

and $C_2\equiv(2,1), r_2=1$

Angle between the two circles is

$$\cos\theta=\frac{-(C_1C_2)^2+(r_1^2+r_2^2)}{2r_1r_2}$$
$$\cos\theta=\frac{-[(2-1)^2+(1-2)^2]+(5-c+1)}{2\sqrt{5-c}\ (1)}$$

$\Rightarrow \quad \cos 60°=\frac{-(1+1)+5-c+1}{2\sqrt{5-c}}$

$\Rightarrow \quad \frac{1}{2}=\frac{-c+4}{2\sqrt{5-c}}$

$\Rightarrow \quad \sqrt{5-c}=-c+4$

On squaring both sides, we get

$$5-c=c^2-8c+16$$

$\Rightarrow \quad c^2-7c+11=0$

$\therefore \quad c=\frac{7\pm\sqrt{49-44}}{2}=\frac{7\pm\sqrt{5}}{2}$

54. ***(a)*** Let the equation of circle be

$$S\equiv x^2+y^2+2gx+2fy+c=0$$

Since, this circle cuts orthogonally to the circles

$$x^2+y^2-4x-2y+4=0$$
$$x^2+y^2-2x-4y+1=0$$

and $x^2+y^2+4x+2y+1=0$, respectively.

$\therefore \quad 2g(2)+2f(1)=c+4$ …(i)

$2g(1)+2f(2)=c+1$ …(ii)

and $2g(-2)+2f(-1)=1+c$ …(iii)

On solving Eqs. (i), (ii) and (iii), we get

$$g=\frac{3}{4}, f=-\frac{3}{4}, c=\frac{-10}{4}$$

$\therefore$ Equation of circle

$$S\equiv x^2+y^2+\frac{3}{2}x-\frac{3}{2}y-\frac{10}{4}=0$$

$\therefore$ Radius of circle

$$S=\sqrt{\left(\frac{3}{4}\right)^2+\left(\frac{-3}{4}\right)^2+\frac{10}{4}}=\sqrt{\frac{9}{16}+\frac{9}{16}+\frac{10}{4}}=\sqrt{\frac{58}{16}}=\sqrt{\frac{29}{8}}$$

55. ***(b)*** **Key Idea** Distance between the vertex and focus of a parabola is one-fourth of length of its latusrectum.

Equation of parabola is

$$x^2-2x+3y-2=0$$

$\Rightarrow \quad x^2-2x+1=-3y+2+1$

$\Rightarrow \quad (x-1)^2=-3(y-1)$

Distance between the vertex and focus of parabola

$$=\frac{1}{4}\times\text{Length of latusrectum}=\frac{1}{4}\times3=\frac{3}{4}$$

56. ***(c)*** **Key Idea** The coordinates of the end points of a focal chord of the parabola $y^2=4ax$ are $(at^2, 2at)$ and $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

(x_1, y_1) and (x_2, y_2) are the end points of a focal chord of the parabola $y^2=5x$.

Let $\quad x_1=\frac{5}{4}t_1^2$ and $y_1=\frac{5}{2}t_1$

$\therefore \quad x_2=\frac{5}{4}\left(\frac{1}{t_1^2}\right)$ and $y_2=\frac{5}{2}\left(\frac{-1}{t_1}\right)$

$[\because t_1t_2=-1$ as t_1 and t_2 are the end points of a focal chord]

Now, $x_1x_2 = \frac{25}{16}$

and $y_1y_2 = -\frac{25}{4}$

$\therefore\ 4x_1x_2 + y_1y_2 = 4\left(\frac{25}{16}\right) - \frac{25}{4} = \frac{25}{4} - \frac{25}{4} = 0$

57. ***(a)*** We have, $x = 3\cos\theta$ ...(i)

and $y = 4\sin\theta$ ...(ii)

On squaring and adding, we get

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

$\therefore$ Distance between their foci

$$= 2\sqrt{b^2 - a^2} = 2\sqrt{16 - 9} = 2\sqrt{7}$$

58. ***(b)*** Given,

$$9x^2 + 25y^2 - 36x + 50y - 164 = 0$$

$\Rightarrow\ 9x^2 - 36x + 36 + 25y^2 + 50y + 25 = 164 + 36 + 25$

$\Rightarrow\ 9(x^2 - 4x + 4) + 25(y^2 + 2y + 1) = 225$

$\Rightarrow\ 9(x-2)^2 + 25(y+1)^2 = 225$

$\Rightarrow\ \frac{(x-2)^2}{25} + \frac{(y+1)^2}{9} = 1$

Eccentricity of ellipse, $e = \sqrt{1 - \left(\frac{b}{a}\right)^2} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

Equation of latusrectum is

$$x - 2 = \pm ae$$

$\Rightarrow\ x - 2 = \pm 5 \times \frac{4}{5}$

$\Rightarrow\ x - 2 = \pm 4$

$\Rightarrow\ x = \pm 4 + 2$

$\Rightarrow\ x = 6$ and $x = -2$

$\therefore\ x - 6 = 0,\ x + 2 = 0$

59. ***(b)*** Given,

$$y = mx + 2 \text{ and } 4x^2 - 9y^2 = 36$$

Substituting the value of $y = mx + 2$ in equation of hyperbola $4x^2 - 9y^2 = 36$, we get

$$4x^2 - 9(mx + 2)^2 = 36$$

$\Rightarrow\ 4x^2 - 9(m^2x^2 + 4mx + 4) = 36$

$\Rightarrow\ x^2(4 - 9m^2) - 36mx - 72 = 0$

Since, this line is a tangent of hyperbola.

$\therefore\ (36m)^2 + 4 \times 72(4 - 9m^2) = 0$

$\Rightarrow\ 36 \times 36m^2 + 4 \times 36 \times 2(4 - 9m^2) = 0$

$\Rightarrow\ 9m^2 + 8 - 18m^2 = 0$

$\Rightarrow\ 9m^2 = 8$

$\Rightarrow\ m^2 = \frac{8}{9}$

$\therefore\ m = \pm\sqrt{\frac{8}{9}} = \pm\frac{2\sqrt{2}}{3}$

60. ***(b)*** Let $P(2, 3, 4)$, $A(3, -2, 2)$ and $B(6, -17, -4)$.

Let P divides AB in the ratio $k : 1$, then

$$2 = \frac{6k + 3}{k + 1}$$

$\Rightarrow\ 2k + 2 = 6k + 3$

$\Rightarrow\ -4k = 1$

$\Rightarrow\ k = -\frac{1}{4}$

Harmonic conjugate Q divides in the ratio $-k : 1$, i.e. $\frac{1}{4} : 1$

$$Q = \left[\frac{\frac{1}{4}(6) + 3}{\frac{1}{4} + 1}, \frac{\frac{1}{4}(-17) - 2}{\frac{1}{4} + 1}, \frac{\frac{1}{4}(-4) + 2}{\frac{1}{4} + 1}\right]$$

$$= \left(\frac{6 + 12}{5}, \frac{-17 - 8}{5}, \frac{-4 + 8}{5}\right) = \left(\frac{18}{5}, -\frac{25}{5}, \frac{4}{5}\right) = \left(\frac{18}{5}, -5, \frac{4}{5}\right)$$

61. ***(b)*** Let a line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

$\Rightarrow\ 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma + 1 - \sin^2\delta = \frac{4}{3}$

$\therefore\ \sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = 4 - \frac{4}{3} = \frac{8}{3}$

62. ***(c)*** Equation of plane is

$$56x + 4y + 9z = 2016$$

this plane meets the coordinate axes in A, B, C

$\therefore$ Coordinates of $A = \left(\frac{2016}{56}, 0, 0\right)$

Coordinates of $B = \left(0, \frac{2016}{4}, 0\right)$

Coordinates of $C = \left(0, 0, \frac{2016}{9}\right)$

Now, centroid of $\Delta ABC = \left(\frac{2016}{56 \times 3}, \frac{2016}{4 \times 3}, \frac{2016}{9 \times 3}\right) = (12, 168, 224)$

63. ***(a)*** **Key Idea** For a function to be continuous at $x = a$

LHL = RHL = $f(a)$

We have, $f(x) = \begin{cases} 1 - x, & x < 1 \\ (1 - x)(2 - x), & 1 \le x \le 2 \\ 3 - x, & x > 2 \end{cases}$

$\because$ By definition of continuity

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$\Rightarrow\ \lim_{x \to 1^-}(1 - x) = \lim_{x \to 1^+}(1 - x)(2 - x) = (1 - 1)(2 - 1) = 0$

$\therefore\ f(x)$ is continuous at $x = 1$.

Now, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\Rightarrow\ \lim_{x \to 2^-}(1 - x)(2 - x) = \lim_{x \to 2^+}(3 - x) = (1 - 2)(2 - 2)$

$\Rightarrow\ (1 - 2)(2 - 2) = (3 - 2) = 0$

$\Rightarrow\ 0 \neq 1 \neq 0$

$\therefore\ f(x)$ is discontinuous at $x = 2$.

Hence, $f(x)$ is continuous for all real numbers except 2.

64. ***(a)*** We have,

$$\lim_{x \to 0} \frac{6^x - 3^x - 2^x + 1}{x^2}$$

$$= \lim_{x \to 0} \frac{3^x \times 2^x - 3^x - 2^x + 1}{x^2} = \lim_{x \to 0} \frac{3^x(2^x - 1) - 1(2^x - 1)}{x^2}$$

$$= \lim_{x \to 0} \frac{3^x - 1}{x} \times \lim_{x \to 0} \frac{2^x - 1}{x} = (\log_e 2)(\log_e 3)$$

Shortcut Method

$$\lim_{x\to 0}\frac{6^x-3^x-2^x+1}{x^2}=\lim_{x\to 0}\frac{6^x\log 6-3^x\log 3-2^x\log 2}{2x}$$

[using L'Hospital rule]

$$=\lim_{x\to 0}\frac{6^x(\log 6)^2-3^x(\log 3)^2-2^x(\log 2)^2}{2}$$

[using L'Hospital rule]

$$=\frac{(\log 6)^2-(\log 3)^2-(\log 2)^2}{2}$$

$$=\frac{(\log 3+\log 2)^2-(\log 3)^2-(\log 2)^2}{2}$$

$$=\frac{2(\log 3)(\log 2)}{2}=(\log 3)(\log 2)$$

65. ***(a)*** Given,

$$f(x)=\begin{cases}x^2+bx+c, & x<1\\ x, & x\geq 1\end{cases}$$

$f(x)$ is differentiable at $x=1$.

$$\therefore\quad f'(x)=\begin{cases}2x+b, & x<1\\ 1, & x\geq 1\end{cases}$$

Now, $f(x)$ is differentiable at $x=1$.

$\therefore\quad f'(1^-)=f'(1^+)$

$\Rightarrow\quad 2+b=1\Rightarrow b=-1$

$f(x)$ is differentiable at $x=1$.

$\therefore f(x)$ is continuous at $x=1$.

$\Rightarrow\quad \lim_{x\to 1^-}f(x)=\lim_{x\to 1^+}f(x)=f(1)$

$\Rightarrow\quad \lim_{x\to 1^-}(x^2+bx+c)=\lim_{x\to 1^+}x=1$

$\Rightarrow\quad 1+b+c=1$

$\Rightarrow\quad 1-1+c=1\Rightarrow c=1$

Hence, $b-c=-1-1=-2$

66. ***(d)*** $x=a$ is a root of multiplicity two of a polynomial equation $f(x)=0$.

Let $\quad f(x)=(x-a)^2g(x)$

$\Rightarrow\quad f'(x)=2(x-a)g(x)+(x-a)^2g'(x)$

Now, $\quad f''(x)=2g(x)+2(x-a)g'(x)$

$$+2(x-a)g'(x)+(x-a)^2g''(x)$$

$$=2g(x)+4(x-a)g'(x)+(x-a)^2g''(x)$$

$\Rightarrow\quad f'(a)=2(a-a)g(a)+(a-a)^2g'(a)=0$

$\Rightarrow\quad f''(a)=2g(a)+4(a-a)g'(a)+(a-a)^2g''(a)=2g(a)$

$\therefore\quad f(a)=f'(a)=0, f''(a)\neq 0$

67. ***(c)*** Given, $y=\log_2(\log_2 x)$

$\Rightarrow\quad y=\log_2\left(\frac{\log x}{\log 2}\right)\qquad\left[\because \log_a b=\frac{\log b}{\log a}\right]$

$\Rightarrow\quad y=\frac{\log\left(\frac{\log x}{\log 2}\right)}{\log 2}$

$\Rightarrow\quad y=\frac{\log(\log x)-\log(\log 2)}{\log 2}\qquad\left[\because \log\frac{a}{b}=\log a-\log b\right]$

$$\therefore\frac{dy}{dx}=\frac{1}{\log 2}\left[\frac{1}{\log_e x}\times\frac{1}{x}-0\right]=\frac{1}{\log 2\times\log_e x\times x}=\frac{1}{(x\log_e x)\log_e 2}$$

68. ***(b)*** **Key Idea** Angle between two curves is given by $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$ where, m_1 and m_2 are the slopes of tangents of two curves at their intersecting point.

Given, $\quad y^2+x^2=a^2\sqrt{2}\qquad\ldots$(i)

and $\quad x^2-y^2=a^2\qquad\ldots$(ii)

On solving Eqs. (i) and (ii), we get

$$x=a\sqrt{\frac{\sqrt{2}+1}{2}},\ y=a\sqrt{\frac{\sqrt{2}-1}{2}}$$

Now, $\quad y^2+x^2=a^2\sqrt{2}$

$$2y\left(\frac{dy}{dx}\right)_1+2x=0\Rightarrow\left(\frac{dy}{dx}\right)_1=-\frac{x}{y}$$

$$\therefore\quad\left(\frac{dy}{dx}\right)_1=\frac{-a\sqrt{\frac{\sqrt{2}+1}{2}}}{a\sqrt{\frac{\sqrt{2}-1}{2}}}=-\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}$$

and $\quad x^2-y^2=a^2$

On differentiating $\left(\frac{dy}{dx}\right)_2=\frac{x}{y}=\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}$

$$\text{Then, }\tan\theta=\frac{\left(\frac{dy}{dx}\right)_1-\left(\frac{dy}{dx}\right)_2}{1+\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2}=\frac{-\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}-\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}}{1+\left(-\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}\right)\left(\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}\right)}$$

$$=\frac{-2\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}}{1-\frac{\sqrt{2}+1}{\sqrt{2}-1}}=\frac{-2\sqrt{(\sqrt{2}+1)(\sqrt{2}-1)}}{\sqrt{2}-1-\sqrt{2}-1}=\frac{-2\sqrt{2}-1}{-2}=1$$

$$\therefore\quad\theta=\tan^{-1}1=\frac{\pi}{4}$$

69. ***(b)*** Let $y=\tan A\tan B$

$$y=\tan A\tan\left(\frac{\pi}{3}-A\right)\qquad\left[\because A+B=\frac{\pi}{3}\right]$$

$$\Rightarrow\quad\frac{dy}{dA}=\sec^2 A\tan\left(\frac{\pi}{3}-A\right)-\sec^2\left(\frac{\pi}{3}-A\right)\tan A$$

For maxima or minima, put $\frac{dy}{dA}=0$

$$\therefore\quad\sec^2 A\tan\left(\frac{\pi}{3}-A\right)-\sec^2\left(\frac{\pi}{3}-A\right)\tan A=0$$

$$\Rightarrow\quad\sec^2 A\tan\left(\frac{\pi}{3}-A\right)=\tan A\sec^2\left(\frac{\pi}{3}-A\right)$$

$$\Rightarrow\quad(1+\tan^2 A)\tan\left(\frac{\pi}{3}-A\right)=\tan A\left[1+\tan^2\left(\frac{\pi}{3}-A\right)\right]$$

$$\Rightarrow\quad\tan\left(\frac{\pi}{3}-A\right)+\tan^2 A\tan\left(\frac{\pi}{3}-A\right)$$

$$=\tan A+\tan A\tan^2\left(\frac{\pi}{3}-A\right)$$

$$\Rightarrow\quad\left[\tan\left(\frac{\pi}{3}-A\right)-\tan A\right]\left[1-\tan A\tan\left(\frac{\pi}{3}-A\right)\right]=0$$

$$\Rightarrow\quad\tan\left(\frac{\pi}{3}-A\right)=\tan A$$

$\Rightarrow \quad \frac{\pi}{3} - A = A \Rightarrow 2A = \frac{\pi}{3}$

$\Rightarrow \quad A = \frac{\pi}{6}$

$\therefore \quad A = B = \frac{\pi}{6}$

Maximum value of $\tan A \tan B = \tan\frac{\pi}{6}\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = \frac{1}{3}$

70. ***(c)*** Let $P\left(t, \frac{-1}{t}\right)$ be a point on $xy = -1$.

Equation of tangent at P is

$$\frac{1}{2}\left[x\left(\frac{-1}{t}\right) + ty\right] = -1$$

$\Rightarrow \quad -x + t^2 y = -2t$

$\Rightarrow \quad y = \frac{x}{t^2} + \frac{2}{t}$ is also at tangent of $y^2 = 8x$, i.e.

$$c = \frac{a}{m}$$

$\Rightarrow \quad \frac{2}{t} = \frac{2}{1/t^2}$

$\Rightarrow \quad t^3 = 1 \quad \Rightarrow \quad t = 1$

Hence, the equation of common tangent is

$$y = x + 2$$

71. ***(a)*** Given,

$$f(x) = x(x+3)(x-2) = x(x^2 + x - 6)$$

$\Rightarrow \quad f(x) = x^3 + x^2 - 6x$

$\Rightarrow \quad f'(x) = 3x^2 + 2x - 6$

Now, $\quad f'(c) = 3c^2 + 2c - 6$

$\Rightarrow \quad 3c^2 + 2c - 6 = 10 \qquad [\because f'(c) = 10]$

$\Rightarrow \quad 3c^2 + 2c - 16 = 0$

$\Rightarrow \quad 3c^2 + 8c - 6c - 16 = 0$

$\Rightarrow \quad (3c + 8)(c - 2) = 0$

$\therefore \quad c = -\frac{8}{3}$ or $c = 2$

$\Rightarrow \quad c = 2$ as $c = \frac{-8}{3} \notin [-1, 4]$

72. ***(d)*** Here,

$$\int x^3 e^{5x} dx = x^3 \int e^{5x} dx - \int \left\{\frac{dx^3}{dx} \int e^{5x} dx\right\} dx$$

[by using integration by parts]

$$= \frac{x^3 e^{5x}}{5} - \int \frac{3x^2 e^{5x}}{5} dx + C$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3}{5}x^2 \int e^{5x} dx + \frac{3}{5}\int \left\{\frac{dx^2}{dx}\int e^{5x} dx\right\} dx + C$$

[again by using integration by parts]

$$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25}\int x e^{5x} dx$$

Again, by using integration by parts

$$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x}{25}\int e^{5x} dx - \frac{6}{25}\int \left\{\frac{dx}{dx}\int e^{5x} dx\right\} dx + C$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x e^{5x}}{125} - \frac{6}{125}\int e^{5x} dx + C$$

$$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x e^{5x}}{125} - \frac{6e^{5x}}{625} + C$$

$$= \frac{e^{5x}}{625}[125x^3 - 75x^2 + 30x - 6] + C$$

$\therefore \quad f(x) = 5^3 x^3 - 75x^2 + 30x - 6$

73. ***(c)*** Let $I = \int \frac{x}{(x^2 + 2x + 2)^2} dx$

$$I = \int \frac{x}{[(x+1)^2 + 1]^2} dx$$

Put $x + 1 = \tan\theta$

$\Rightarrow \quad dx = \sec^2\theta d\theta$

$$\therefore I = \int \frac{(\tan\theta - 1)\sec^2\theta}{(\tan^2\theta + 1)^2} d\theta = \int \frac{(\tan\theta - 1)\sec^2\theta}{\sec^4\theta} d\theta$$

$$= \int \frac{\tan\theta - 1}{\sec^2\theta} d\theta = \int \left(\frac{\tan\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta}\right) d\theta$$

$$= \int (\sin\theta\cos\theta - \cos^2\theta) d\theta$$

$$= \frac{1}{2}\int (2\sin\theta\cos\theta - 2\cos^2\theta) d\theta = \frac{1}{2}\int (\sin 2\theta - 1 - \cos 2\theta) d\theta$$

$$= \frac{1}{2}\left[\frac{-\cos 2\theta}{2} - \theta - \frac{\sin 2\theta}{2}\right] + C = -\frac{1}{4}[\cos 2\theta + \sin 2\theta] - \frac{1}{2}\theta + C$$

$$= -\frac{1}{4}\left[\frac{1 - \tan^2\theta}{1 + \tan^2\theta} + \frac{2\tan\theta}{1 + \tan^2\theta}\right] - \frac{1}{2}\tan^{-1}(x+1) + C$$

$$= -\frac{1}{4}\left[\frac{1 - (x+1)^2 + 2(x+1)}{1 + (x+1)^2}\right] - \frac{1}{2}\tan^{-1}(x+1) + C$$

$$= \frac{1}{4}\left(\frac{x^2 - 2}{x^2 + 2x + 2}\right) - \frac{1}{2}\tan^{-1}(x+1) + C$$

74. ***(c)*** Let $I = \int \log(a^2 + x^2) dx$

By using integration by parts, we get

$$= \log(a^2 + x^2)\int dx - \int \left\{\frac{d(\log(x^2 + a^2))}{dx}\int dx\right\} dx + C$$

$$= x\log(a^2 + x^2) - \int \frac{2x}{x^2 + a^2} \times x\, dx + C$$

$$= x\log(a^2 + x^2) - 2\int \frac{x^2}{x^2 + a^2} dx + C$$

$$= x\log(a^2 + x^2) - 2\int \frac{x^2 + a^2}{x^2 + a^2} dx + 2\int \frac{a^2}{x^2 + a^2} dx$$

$$= x\log(a^2 + x^2) - 2x + \frac{2a^2}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$= x\log(a^2 + x^2) - 2x + 2a\tan^{-1}\left(\frac{x}{a}\right) + C$$

$\therefore \quad h(x) = x\log(a^2 + x^2) - 2x + 2a\tan^{-1}\left(\frac{x}{a}\right)$

75. ***(a)*** We have,

$$\int (\log x)^5 dx = x[A(\log x)^5 + B(\log x)^4 + C(\log x)^3 + D(\log x)^2 + E(\log x) + F] + C$$

Let $\quad I = \int (\log x)^5 dx$

Put $\quad \log x = t \Rightarrow x = e^t$

$\Rightarrow \quad dx = e^t dt$

$\therefore \quad I = \int t^5 e^t dt = e^t [t^5 - 5t^4 + 20t^3 - 60t^2 + 120t - 120] + C$

$$= x[(\log x)^5 - 5(\log x)^4 + 20(\log x)^3 - 60(\log x)^2 + 120(\log x) - 120] + C$$

$\therefore A + B + C + D + E + F = 1 - 5 + 20 - 60 + 120 - 120 = -44$

76. ***(d)*** For point of intersection, equate both equations.

$$\frac{x^2}{4a}=\frac{8a^3}{x^2+4a^2}$$

$\Rightarrow \quad x^2(x^2+4a^2)=4a(8a^3)$

$\Rightarrow \quad x^4+4a^2x^2=32a^4$

$\Rightarrow \quad x^4+4a^2x^2-32a^4=0$

$\Rightarrow \quad x^4+8a^2x^2-4a^2x^2-32a^4=0$

$\Rightarrow \quad x^2(x^2+8a^2)-4a^2(x^2+8a^2)=0$

$\Rightarrow \quad (x^2-4a^2)(x^2+8a^2)=0$

$\Rightarrow \quad x^2-4a^2=0 \text{ or } x^2+8a^2=0$

$\therefore \quad x^2=4a^2 \qquad [\because x^2=-8a^2 \text{ is not possible}]$

$\Rightarrow \quad x=\pm 2a$

$\Rightarrow \quad x=2a$

Hence, we take limits from 0 to $2a$.

$\therefore$ Area between given curves

$$=2\times\int_0^{2a}\left[\frac{8a^3}{(x^2+4a^2)}-\frac{x^2}{4a}\right]dx=2\times\left[\int_0^{2a}\frac{8a^3}{(x^2+4a^2)}dx-\int_0^{2a}\frac{x^2}{4a}dx\right]$$

$$=2\times\left[8a^3\times\int_0^{2a}\frac{1}{x^2+(2a)^2}dx-\frac{1}{4a}\int_0^{2a}x^2dx\right]$$

$$=2\times\left[8a^3\left[\frac{1}{2a}\tan^{-1}\left(\frac{x}{2a}\right)\right]_0^{2a}-\frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{2a}\right]$$

$$=2\times\left[\frac{8a^3}{2a}\left[\tan^{-1}\left(\frac{2a}{2a}\right)-\tan^{-1}\left(\frac{0}{2a}\right)\right]-\frac{1}{4a}\left[\frac{(2a)^3}{3}-\frac{(0)^3}{3}\right]\right]$$

$$=2\times\left[4a^2[\tan^{-1}(1)-\tan^{-1}(0)]-\frac{1}{4a}\left(\frac{8a^3}{3}-0\right)\right]$$

$$=2\times\left[4a^2\left(\frac{\pi}{4}-0\right)-\frac{1}{4a}\left(\frac{8a^3}{3}\right)\right]$$

$$=2\times\left[4a^2\times\frac{\pi}{4}-\frac{1}{4a}\times\frac{8a^3}{3}\right]=2\times\left[a^2\pi-\frac{2a^2}{3}\right]$$

$$=2\left[a^2\left(\pi-\frac{2}{3}\right)\right]=2a^2\left(\pi-\frac{2}{3}\right)=a^2\left(2\pi-\frac{4}{3}\right)$$

Hence, the required area between the parabola $y=\frac{x^2}{4a}$ and the curve $y=\frac{8a^3}{(x^2+4a^2)}$ is $a^2\left(2\pi-\frac{4}{3}\right)$

77. ***(b)*** We have,

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\ldots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{\sqrt{1-\left(\frac{1}{n}\right)^2}}+\frac{1}{\sqrt{1-\left(\frac{2}{n}\right)^2}}+\ldots+\frac{1}{\sqrt{1-\left(\frac{n-1}{n}\right)^2}}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n-1}\frac{1}{\sqrt{1-\left(\frac{r}{n}\right)^2}}=\int_0^1\frac{dx}{\sqrt{1-x^2}}=[\sin^{-1}x]_0^1$$

$$=\sin^{-1}1-\sin^{-1}0=\frac{\pi}{2}$$

78. ***(d)*** Let $I=\int_{-\pi/4}^{\pi/4}\frac{x+\frac{\pi}{4}}{2-\cos 2x}dx$

$$=\int_{-\pi/4}^{\pi/4}\frac{x\,dx}{2-\cos 2x}+\frac{\pi}{4}\int_{-\pi/4}^{\pi/4}\frac{dx}{2-\cos 2x}$$

$$=0+2\times\frac{\pi}{4}\int_0^{\pi/4}\frac{dx}{2-\cos 2x} \qquad \left[\because \frac{x}{2-\cos 2x}\text{ is odd function}\right]$$

$$=\frac{\pi}{2}\int_0^{\pi/4}\frac{(1+\tan^2 x)}{2+2\tan^2 x-1+\tan^2 x}dx$$

$$=\frac{\pi}{2}\int_0^{\pi/4}\frac{\sec^2 x}{1+3\tan^2 x}dx$$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$\therefore \quad \frac{\pi}{2}\int_0^1\frac{dt}{1+3t^2}=\frac{\pi}{2}\times\frac{1}{\sqrt{3}}[\tan^{-1}\sqrt{3}t]_0^1$

$$=\frac{\pi}{2\sqrt{3}}[\tan^{-1}\sqrt{3}-\tan^{-1}0]=\frac{\pi}{2\sqrt{3}}\left(\frac{\pi}{3}\right)=\frac{\pi^2}{6\sqrt{3}}$$

79. ***(c)*** We have,

$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{\tan^{-1}y}}{1+y^2}$$

This equation is form of linear differential equation

$$\frac{dx}{dy}+Px=Q$$

Where, $P=\frac{1}{1+y^2}, Q=\frac{e^{\tan^{-1}y}}{1+y^2}$

Now, $IF=e^{\int P dy}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$

$\therefore$ Solution of differential equation is

$$x\times IF=\int Q\times IF\,dy+C$$

$\Rightarrow \quad xe^{\tan^{-1}y}=\int\frac{e^{2\tan^{-1}y}}{1+y^2}dy+C$

$\Rightarrow \quad xe^{\tan^{-1}y}=\frac{e^{2\tan^{-1}y}}{2}+C \quad \Rightarrow \quad 2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+C$

80. ***(d)*** We have, $(2x-4y+3)\frac{dy}{dx}+(x-2y+1)=0$

$\Rightarrow \quad \frac{dy}{dx}=\frac{-(x-2y+1)}{2(x-2y)+3}$

Put $\quad x-2y=v \Rightarrow 1-2\frac{dy}{dx}=\frac{dv}{dx}$

$\therefore \quad \frac{1}{2}\left(1-\frac{dv}{dx}\right)=-\left(\frac{v+1}{2v+3}\right)$

$\Rightarrow \quad \frac{dv}{dx}=1+\frac{2v+2}{2v+3} \qquad \Rightarrow \quad \frac{2v+3}{4v+5}dv=dx$

$\Rightarrow \quad \frac{1}{2}\int\frac{4v+6}{4v+5}dv=\int dx$

$\Rightarrow \quad \frac{1}{2}\int\left(\frac{4v+5}{4v+5}+\frac{1}{4v+5}\right)dv=\int dx$

$\Rightarrow \quad \frac{1}{2}v+\frac{1}{2\times 4}\log[4v+5]=x+C$

$\Rightarrow \quad 4v+\log[4v+5]=8x+C$

$\Rightarrow 4(x-2y)+\log[4(x-2y)+5]=8x+C$

$\Rightarrow \log[4(x-2y)+5]=4(x+2y)+C$

CONDITIONAL PROBABILITY

Probability is one of the most important topic from competitive point of view. In this article, we will discuss about Conditional probability, Total probability and Baye's theorem. Topics covered in this article are frequently asked in both JEE Main and JEE Advanced.

Conditional Theorem

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of event A under the condition that B has already occurred and $P(B) \neq 0$, is called the conditional probability and it is denoted by $P\left(\frac{A}{B}\right)$.

Thus, we have $P\left(\frac{A}{B}\right)$ = Probability of occurrence of A given that B has already occurred

Or $$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

For $P\left(\frac{A}{B}\right)$, B act as sample space and common elements between A and B are favourable events.

POINTS TO REMEMBER

1. Let A and B be two events of a sample space S of an experiment, then we have $P\left(\frac{A}{S}\right)=P\left(\frac{B}{B}\right)=1$.
2. If A and B are any two events of a sample space S and E is the event of S such that $P(E)\neq 0$, then
$$P\left(\frac{A\cup B}{E}\right)=P\left(\frac{A}{E}\right)+P\left(\frac{B}{E}\right)-P\left(\frac{A\cap B}{E}\right)$$
3. $P\left(\frac{\bar{E}_1}{E_2}\right)=1-P\left(\frac{E_1}{E_2}\right)$, where $P(\bar{E}_1)$ is probability of not occurring 'E_1'.

Example 1 A sample consists of six points. $S=\{(i,j)|\, i=1,2;\ j=1,2,3\}$. Suppose $P(i,j)=\frac{k}{i+j}$, let $A=\{(i,j)|\, i+j=4\}$ and $B=\{(i,j)|\, j=2\}$, then $P(B/A)$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

Sol. ***(a)*** We have,
$$P\left(\frac{B}{A}\right)=\frac{P(B\cap A)}{P(A)}=\frac{P(2,2)}{P\{(1,3),(2,2)\}}$$
$$=\frac{\frac{k}{2+2}}{\frac{k}{1+3}+\frac{k}{2+2}}=\frac{\frac{k}{4}}{\frac{k}{4}+\frac{k}{4}}=\frac{1}{2}$$

Example 2 Let X and Y be two events such that $P\left(\frac{X}{Y}\right)=\frac{1}{2}$, $P\left(\frac{Y}{X}\right)=\frac{1}{3}$ and $P(X\cap Y)=\frac{1}{6}$.

Which of the following is incorrect? **[IIT JEE 2012]**

(a) $P(X\cup Y)=\frac{2}{3}$ (b) X and Y are independent

(c) $P(X^c\cap Y)=\frac{1}{6}$ (d) X and Y are not independent

Sol. ***(d)*** We have,
$$P\left(\frac{X}{Y}\right)=\frac{1}{2}, P\left(\frac{Y}{X}\right)=\frac{1}{3} \text{ and } P(X\cap Y)=\frac{1}{6}$$
$$\therefore \quad P(X)=\frac{P(X\cap Y)}{P\left(\frac{Y}{X}\right)}=\frac{1/6}{1/3}=\frac{1}{2}$$
$$P(Y)=\frac{P(X\cap Y)}{P\left(\frac{X}{Y}\right)}=\frac{1/6}{1/2}=\frac{1}{3}$$
$$\therefore \quad P(X)\cdot P(Y)=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$$
Clearly, $P(X\cap Y)=P(X)\cdot P(Y)$
So, X and Y are independent events.
$$\therefore \quad P(X\cup Y)=1-P(\bar{X})P(\bar{Y})=1-\frac{1}{2}\times\frac{2}{3}=\frac{2}{3}$$
and $$P(X^c\cap Y)=P(Y)-P(X\cap Y)=\frac{1}{3}-\frac{1}{6}=\frac{1}{6}$$
Hence, option (d) is incorrect.

Multiplication Theorem

Le A and B be two events associated with a sample space S. $A\cap B$ denotes the simultaneous occurrence of the events A and B. The event $A\cap B$ can also be denoted by AB.

Now, according to conditional theorem
$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}$$
$$\Rightarrow \quad P(A\cap B)=P(B)\times P\left(\frac{A}{B}\right), P(B)\neq 0$$

This result is known as multiplication theorem.

In general,
$$P\left(\bigcap_{i=1}^{n}A_i\right)=P(A_1)\,P\left(\frac{A_2}{A_1}\right)\cdot P\left(\frac{A_3}{A_1\cap A_2}\right)$$
$$\ldots P\left(\frac{A_n}{A_1\cap A_2\cap A_3\cap\ldots\cap A_{n-1}}\right)$$

Example 3 A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one and examined. The ones examined are not put back. The probability that 9th one examined is the last defective, is

(a) $\frac{{}^4C_3\times{}^{11}C_5}{{}^{15}C_6}$ (b) $\frac{{}^4C_3\times{}^{11}C_5}{{}^{15}C_8}\times\frac{1}{7}$

(c) $\frac{{}^{11}C_5}{{}^{15}C_8}\times\frac{1}{7}$ (d) $\frac{{}^4C_3}{{}^{11}C_5}\times\frac{1}{7}$

Sol. ***(b)*** Let A and B be two events defined by
$A=$ Getting exactly 3 defectives in 8 selection
$B=$ 9th record player is defective
Required probability $=P(A\cap B)=P(A)\,P\left(\frac{B}{A}\right)$ ...(i)
Now, $$P(A)=\frac{{}^4C_3\times{}^{11}C_5}{{}^{15}C_8}$$
and $P\left(\frac{B}{A}\right)=$ Probability that the 9th examined record player is defective given that there were 3 defectives in the first 8 pieces examined
$$\Rightarrow \quad P\left(\frac{B}{A}\right)=\frac{1}{7}$$
$\therefore$ Required probability $=\frac{{}^4C_3\times{}^{11}C_5}{{}^{15}C_8}\times\frac{1}{7}$

Example 4 Cards are drawn one by one without replacement from a pack of 52 cards. The probability that 10 cards will precede the first ace is

(a) $\frac{241}{1456}$ (b) $\frac{82}{1547}$ (c) $\frac{451}{884}$ (d) None of these

Sol. ***(b)*** Let A be the event drawing 9 cards which are not ace and B be drawing an ace card. Therefore, required probability is

$$P(A \cap B) = P(A) \times P\left(\frac{B}{A}\right)$$

Now, there are four aces and 48 other cards. Hence, $P(A) = \frac{{}^{48}C_9}{{}^{52}C_9}$

After having drawn 9 non-ace cards, the 10th card must be ace. Hence,

$$P(B) = \frac{{}^{4}C_1}{{}^{43}C_1} = \frac{4}{43}$$

Hence, $P(A \cap B) = \frac{{}^{48}C_9}{{}^{52}C_9} \times \frac{4}{43} = \frac{82}{1547}$

Independent Events

Two events A and B are said to be an independent events, if probability of occurrence of one of them is not affected by occurrence of the other.

If two events are independent, then

$P\left(\frac{A}{B}\right) = P(A)$, provided $P(B) \neq 0$

and $P\left(\frac{B}{A}\right) = P(B)$, provided $P(A) \neq 0$

Also, using multiplication theorem

$$P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right)$$

$$\Rightarrow \quad P(A \cap B) = P(A) \cdot P(B) \qquad \left[\because P\left(\frac{B}{A}\right) = P(B)\right]$$

Hence, we can conclude that events A and B associated to same random experiment are independent, if

$$P(A \cap B) = P(A) \cdot P(B)$$

POINTS TO REMEMBER

1. Three events A, B and C are said to be mutually independent, if $P(A \cap B) = P(A) \cdot P(B)$, $P(A \cap C) = P(A) \cdot P(C)$ $P(B \cap C) = P(B) \cdot P(C)$ and $P(A \cap B \cap C) = P(A)P(B)P(C)$
   If atleast one of the above is not true, then for the three events, we say that the events are not independent.
2. The events A and ϕ (empty) are independent.
3. The events A and S (sample space) are independent.
4. If A and B are independent, then
   (i) A and B' are independent events.
   (ii) A' and B are independent events.
   (iii) A' and B' are independent events.
5. If A and B are two independent events. Then, the probability of occurrence of atleast one of A and B is given by $1 - P(A')P(B')$ i.e. $P(A \cup B) = 1 - P(A')P(B')$

Example 5 If A and B are independent events of a random experiment such that $P(A \cap B) = \frac{1}{6}$ and $P(\bar{A} \cap \bar{B}) = \frac{1}{3}$, then $P(A)$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{2}{3}$

Sol. ***(b)*** We have,

$$P(A \cap B) = \frac{1}{6} \quad \text{and} \quad P(\bar{A} \cap \bar{B}) = \frac{1}{3}$$

$$\Rightarrow \quad P(A)P(B) = \frac{1}{6} \quad \text{and} \quad P(\bar{A})P(\bar{B}) = \frac{1}{3}$$

[as A and B are independent events]

$$\Rightarrow \quad xy = \frac{1}{6} \quad \text{and} \quad (1-x)(1-y) = \frac{1}{3}$$

[$\because P(A) = x$ and $P(B) = y$]

$$\Rightarrow \quad xy = \frac{1}{6} \quad \text{and} \quad 1 - x - y + xy = \frac{1}{3}$$

$$\Rightarrow \quad xy = \frac{1}{6} \quad \text{and} \quad 1 - x - y + \frac{1}{6} = \frac{1}{3}$$

$$\Rightarrow \quad xy = \frac{1}{6} \quad \text{and} \quad x + y = \frac{5}{6}$$

$$\Rightarrow \quad x = \frac{1}{2} \quad \text{and} \quad y = \frac{1}{3}$$

$$\text{or} \quad x = \frac{1}{3} \quad \text{and} \quad y = \frac{1}{2}$$

Example 6 If odds against solving a question by three students are 2 : 1, 5 : 2 and 5 : 3 respectively, then probability that the question is solved only by one student is

(a) $\frac{31}{56}$ (b) $\frac{24}{56}$
(c) $\frac{25}{56}$ (d) None of these

Sol. ***(c)*** The probabilities of solving the question by these three students are $\frac{1}{3}, \frac{2}{7}$ and $\frac{3}{8}$, respectively. Therefore,

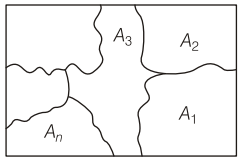
$$P(A) = \frac{1}{3}, \quad P(B) = \frac{2}{7}, \quad P(C) = \frac{3}{8}$$

Then, probability of question solved by only one student is

$$= P(A\bar{B}\bar{C} \text{ or } \bar{A}B\bar{C} \text{ or } \bar{A}\bar{B}C)$$
$$= P(A)P(\bar{B})P(\bar{C}) + P(\bar{A})P(B)P(\bar{C}) + P(\bar{A})P(\bar{B})P(C)$$
$$= \frac{1}{3} \times \frac{5}{7} \times \frac{5}{8} + \frac{2}{3} \times \frac{2}{7} \times \frac{5}{8} + \frac{2}{3} \times \frac{5}{7} \times \frac{3}{8}$$
$$= \frac{25 + 20 + 30}{168} = \frac{25}{56}$$

Partition of a Sample Space

Consider, a sample space S. Let $A_1, A_2, \ldots, A_n$ be the set of mutually exclusive and exhaustive events of sample space S.



These events $A_1, A_2, \ldots, A_n$ are said to partition the sample space S. We have, $A_i \cap A_j = \phi$ for $i \neq j, 1 \leq i, j \leq n$

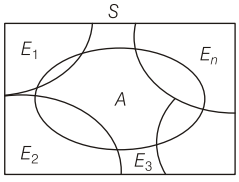
and $\sum_{i=1}^{n} P(A_i) = 1$

Theorem of Total Probability

Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or .... or E_n, then

$$P(A) = P(E_1)\,P\left(\frac{A}{E_1}\right) + P(E_2)\,P\left(\frac{A}{E_2}\right) + \ldots + P(E_n)\,P\left(\frac{A}{E_n}\right)$$

or $$P(A) = \sum_{i=1}^{n} P(E_i)\,P\left(\frac{A}{E_i}\right)$$



Example 7 If from each of the three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn at random, then the probability that 2 white and 1 black ball will be drawn is

[IIT JEE 1998]

(a) $\frac{13}{32}$ (b) $\frac{1}{4}$ (c) $\frac{1}{32}$ (d) $\frac{3}{10}$

Sol. ***(a)*** The contents of three boxes are

I.	3 W	1B
II.	2 W	2B
III.	1 W	3B

Let $W_i\,(i = 1, 2, 3)$ be the event of drawing a white ball from ith box and $B_i\,(i = 1, 2, 3)$ be the event of drawing a black ball from ith box. Then, required probability

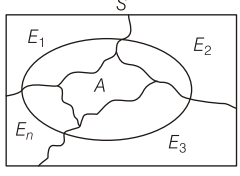
$$= P\{(W_1 \cap W_2 \cap B_3) \cup (W_1 \cap B_2 \cap W_3) \cup (B_1 \cap W_2 \cap W_3)\}$$
$$= P(W_1 \cap W_2 \cap B_3) + P(W_1 \cap B_2 \cap W_3) + P(B_1 \cap W_2 \cap W_3)$$
$$= P(W_1)P(W_2)P(B_3) + P(W_1)P(B_2)P(W_3) + P(B_1)P(W_2)P(W_3)$$
$$= \frac{3}{4}\times\frac{2}{4}\times\frac{3}{4} + \frac{3}{4}\times\frac{2}{4}\times\frac{1}{4} + \frac{1}{4}\times\frac{2}{4}\times\frac{1}{4} = \frac{26}{64} = \frac{13}{32}$$

Baye's Theorem

If $E_1, E_2, E_3, \ldots, E_n$ be n mutually exclusive and exhaustive events and A is an event which occurs together with either of E_i, i.e. if $E_1, E_2, \ldots, E_n$ form a partition of the sample space S and A be any event, then

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i)\,P\left(\frac{A}{E_i}\right)}{P(E_1)\,P\left(\frac{A}{E_1}\right) + P(E_2)\,P\left(\frac{A}{E_2}\right) + \ldots + P(E_n)\,P\left(\frac{A}{E_n}\right)}$$

or $$P\left(\frac{E_i}{A}\right) = \frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^{n} P(E_i)P\left(\frac{A}{E_i}\right)}$$



Example 8 In an entrance test, there are multiple choice questions. There are four possible answers to each question, of which one is correct. The probability that a student knows the answer to a question is 90%. If he gets the correct answer to a question, then the probability that he was guessing is

(a) 1/36 (b) 1/37 (c) 1/38 (d) 1/39

Sol. ***(b)*** We define the following events

E_1 : He knows the answer
E_2 : He does not know the answer
A : He gets the correct answer

Then, $$P(E_1) = \frac{9}{10},\ P(E_2) = 1 - \frac{9}{10} = \frac{1}{10}$$

$$P\left(\frac{A}{E_1}\right) = 1,\ P\left(\frac{A}{E_2}\right) = \frac{1}{4}$$

Therefore, the required probability is

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)}$$
$$= \frac{\frac{1}{10}\times\frac{1}{4}}{\frac{9}{10}\times 1 + \frac{1}{10}\times\frac{1}{4}} = \frac{1}{37}$$

Example 9 A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, the probability that these 2 balls are drawn from box B_2 is

[JEE (Adv) 2013]

(a) 116/181 (b) 126/181 (c) 65/181 (d) 55/181

Sol. ***(d)*** Let $E_i\,(i = 1, 2, 3)$ denotes the event of selecting ith bag and A denotes the event of drawing one white and one red ball.

Then, $$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P\left(\frac{A}{E_1}\right) = \frac{1}{6}\times\frac{3}{5} + \frac{3}{6}\times\frac{1}{5} = \frac{1}{5}$$

$$P\left(\frac{A}{E_2}\right) = \frac{2}{9}\times\frac{3}{8} + \frac{3}{9}\times\frac{2}{8} = \frac{1}{6}$$

$$P\left(\frac{A}{E_3}\right) = \frac{3}{12}\times\frac{4}{11} + \frac{4}{12}\times\frac{3}{11} = \frac{2}{11}$$

$\therefore$ Required probability $= P\left(\frac{E_2}{A}\right)$

$$= \frac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right) + P(E_3)P\left(\frac{A}{E_3}\right)}$$
$$= \frac{\frac{1}{3}\times\frac{1}{6}}{\frac{1}{3}\times\frac{1}{5} + \frac{1}{3}\times\frac{1}{6} + \frac{1}{3}\times\frac{2}{11}} = \frac{55}{181}$$

Smart Practice

1. If $\bar{E}$ and $\bar{F}$ are the complementary events of events E and F, respectively and if $0 < P(F) < 1$, then

(a) $P\left(\frac{E}{F}\right) + P\left(\frac{\bar{E}}{F}\right) = 1$ (b) $P\left(\frac{E}{F}\right) + P\left(\frac{E}{\bar{F}}\right) = 1$

(c) $P\left(\frac{\bar{E}}{F}\right) + P\left(\frac{E}{\bar{F}}\right) = 1$ (d) $P\left(\frac{E}{\bar{F}}\right) + P\left(\frac{\bar{E}}{\bar{F}}\right) = \frac{1}{2}$

2. The event A is independent to itself if and only if $P(A)$ is

(a) 0 or 1 (b) 1/2 (c) 0 (d) 0, 1/2 or 1

3. The probability of the simultaneous occurrence of two events A and B is p. If the probability that exactly one of A, B, occurs is q, then which of the following alternative is incorrect?

(a) $P(A) + P(B) = 2 + 2q$

(b) $P(\bar{A}) + P(\bar{B}) = 2 - 2p - q$

(c) $P\{(A \cap B)/(A \cup B)\} = \frac{p}{p+q}$

(d) $P(\bar{A} \cap \bar{B}) = 1 - p - q$

4. Two aeroplanes I and II bomb a target in succession. The probabilities of I and II scoring a hit correctly are 0.3 and 0.2, respectively. The second plane will bomb only if the first misses the target. The probability that the target is hit by second plane is

(a) 0.2 (b) 0.7 (c) 0.06 (d) 0.14

5. A bag contains $(n+1)$ coins. It is known that one of these coins shows head on both sides, whereas the other coins are fair. One coin is selected at random and tossed. If the probability that toss results in heads is 7/12, then the value of n is

(a) 3 (b) 4 (c) 5 (d) 6

6. There are n urns each containing $(n+1)$ balls such that the ith urn contains 'i' white balls and $(n+1-i)$ red balls. Let u_i be the event of selecting ith urn, $i = 1, 2, 3, \ldots, n$ and W denotes the event of getting a white ball. If $P(u_i) \propto i$, where $i = 1, 2, 3, \ldots, n$, then $\lim_{n\to\infty} P(W)$ is equal to

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$

7. One Indian and four American men and their wives are to be seated randomly around a circular table. The conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife, is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$

8. A signal which can be green or red with probability 4/5 and 1/5 respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is 3/4. If the signal received at station B is green, then the probability that the original signal was green, is

(a) $\frac{3}{5}$ (b) $\frac{6}{7}$ (c) $\frac{20}{23}$ (d) $\frac{9}{20}$

9. If two events A and B are such that $P(A) = 0.3$; $P(B) = 0.4$; $P(\bar{A} \cap \bar{B}) = 0.5$, then the value of $P\left(\frac{B}{A \cup \bar{B}}\right)$ is

(a) 0.25 (b) 0.30 (c) 0.35 (d) 0.40

10. Suppose families always have one, two or three children, with probabilities 1/4, 1/2 and 1/4, respectively. Assume everyone eventually gets married and has children, then the probability of a couple having exactly four grandchildren is

(a) 1/8 (b)27/128 (c) 8/27 (d) 4/27

11. Each of the n urns contains 4 white and 6 black balls. The $(n+1)$th urn contains 5 white and 5 black balls. One of the $(n+1)$ urns is chosen at random and two balls are drawn from it without replacement. Both the balls turn out to be black. If the probability that the $(n+1)$th urn was chosen to draw the balls is 1/16, then the value of n is

(a) 8 (b) 9 (c) 10 (d) 11

12. Die A has 4 red and 2 white faces, whereas die B has 2 red and 4 white faces. A coin is flipped once. If it shows a head, the game continues by throwing die A, if it shows tail, then die B is to be used. If the probability that die A is used is 32/33 when it is given that red turns up every time in first n throws, then the value of n is

(a) 2 (b) 3 (c) 4 (d) 5

13. All the jacks, queens kings and aces of a regular 52 cards deck are taken out. The 16 cards are thoroughly shuffled and my opponent, a person who always tells the truth, simultaneously draws two cards at random and says, "I hold atleast once ace". The probability that the holds two aces is

(a) 2/8 (b) 4/9 (c) 2/3 (d) 1/9

14. An event X can take place in conjuction with any one of the mutually exclusive and exhaustive events A, B and C. If A, B, C are equiprobable and the probability of X is 5/12 and the probability of X taking place when A has happened is 3/8, while it is 1/4 when B has taken place, then probability of X taking place in conjuction with C is

(a) 5/8 (b) 3/8 (c) 5/24 (d) 3/28

15. On a normal standard die one of the 21 dots from any one of the six faces is removed at random with each dot equally likely to be chosen. If the die is rolled, then the probability that the odd number appears is

(a) $\frac{11}{21}$ (b) $\frac{13}{21}$ (c) $\frac{14}{23}$ (d) $\frac{13}{23}$

16. Let A and B be two independent events.

Statement I If $P(A) = 0.4$ and $P(A \cup \overline{B}) = 0.9$, then $P(B)$ is 1/6.

Statement II If A and B are independent, then $P(A \cap B) = P(A) \cdot P(B)$.

(a) Both Statements are correct
(b) Only Statement I is correct
(c) Only Statement II is correct
(d) Both Statements are incorrect

17. A letter is known to have come either from LONDON or CLIFTON; on the postmark only the two consecutive letters ON are legible. The probability that it came from LONDON is

(a) $\frac{1}{17}$ (b) $\frac{12}{17}$ (c) $\frac{17}{30}$ (d) $\frac{3}{5}$

18. If A and B are two events, then the probability that exactly one of them occurs is not given by

(a) $P(A) + P(B) - 2P(A \cap B)$ (b) $P(A \cap \overline{B}) + P(\overline{A} \cap B)$
(c) $P(A \cup B) - P(A \cap B)$ (d) $P(\overline{A}) + P(\overline{B}) - 2P(\overline{A} \cap \overline{B})$

19. Two numbers are selected randomly from the set $S = \{1, 2, 3, 4, 5, 6\}$ without replacement one by one. The probability that the minimum of the two numbers is less than 4, is

(a) $\frac{1}{15}$ (b) $\frac{14}{15}$ (c) $\frac{1}{5}$ (d) $\frac{4}{5}$

20. Five horses are in a race. Mr. A selects two of the horses at random and bets on them. The probability that Mr. A selected the winning horse is

(a) $\frac{2}{5}$ (b) $\frac{4}{5}$ (c) $\frac{3}{5}$ (d) $\frac{1}{5}$

21. If $A_1, A_2, \ldots, A_n$ are n independent events such that $P(A_i) = \frac{1}{i+1}$; $i = 1, 2, \ldots, n$. The probability that none of $A_1, A_2, \ldots, A_n$ occurs is

(a) $\frac{n}{n+1}$ (b) $\frac{1}{n+1}$ (c) $\frac{1}{n!}$ (d) $\frac{1}{n+2}$

22. If C and D are two events such that $C \subseteq D$ and $P(D) \neq 0$, then the correct statement among the following is

(a) $P(C/D) = \frac{P(C)}{P(D)}$ (b) $P(C/D) = P(C)$
(c) $P(C/D) \geq P(C)$ (d) $P(C/D) < P(C)$

23. One ticket is selected at random from 100 tickets numbered 00, 01, 02, ……, 98, 99. If X and Y denote respectively the sum and the product of the digits on the tickets, then $P(X = 9 / Y = 0)$ equals

(a) $\frac{2}{17}$ (b) $\frac{2}{19}$ (c) $\frac{2}{21}$ (d) $\frac{2}{11}$

24. A and B are two students. Their probabilities of solving a problem correctly are 1/4 and 1/5, respectively. If the probability of their making a common error is 1/40 and they obtain the same answer, then the probability of their answer is correct, is

(a) 1/2 (b) 1/20 (c) 10/13 (d) 13/200

25. If $P(A/B) = P(A/B')$, then

(a) A and B are independent (b) $P(A) = 0$
(c) $P(B) = 0$ (d) None of these

26. For two events A and B it is given that $P(A) = P(A/B) = 1/4$ and $P(B/A) = 1/2$. Then,

(a) A and B are mutually exclusive events
(b) A and B are independent events such that $P(\overline{A}/B) = 3/4$
(c) A and B are independent events such that $P(\overline{A}/B) = 1/2$
(d) A and B are in independent events such that $P(\overline{A}/B) = 3/4$

27. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational atleast two of engines must function. Let X denotes the event that ship is rational and let X_1, X_2 and X_3 denote respectively the events that engines E_1, E_2 and E_3 are functioning. Which is the following is true?

(a) $P(X_1^C/X) = \frac{7}{8}$

(b) P [Exactly two engines of the ship are functioning/X] = $\frac{7}{16}$

(c) $P(X/X_2) = \frac{5}{16}$

(d) $P(X/X_1) = \frac{7}{16}$

28. For any two events A and B in a sample space which one of the following is incorrect?

(a) $P(A/B) \geq \frac{P(A) + P(B) - 1}{P(B)}$, $P(B) \neq 0$, is always true
(b) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$ does not hold
(c) $P(A \cup B) = 1 - P(\overline{A})P(\overline{B})$, if A and B are independent
(d) $P(A \cup B) = 1 - P(\overline{A})P(\overline{B})$, if A and B are disjoint

29. The probability that the 13th day of a randomly selected month is a Friday, is

(a) $\frac{1}{12}$ (b) $\frac{1}{7}$
(c) $\frac{1}{84}$ (d) None of these

30. The probability that a teacher will give a surprise test during any class meeting is 1/5. If a student is absent on 2 days, then the probability that the student will miss atleast one test, is

(a) 4/5 (b) 2/5 (c) 7/5 (d) 9/25

Answers with Explanation

1. ***(a)*** We know that,

$$P\left(\frac{E}{F}\right)+P\left(\frac{\bar{E}}{F}\right)=\frac{P(E\cap F)+P(\bar{E}\cap F)}{P(F)}$$

$$=\frac{P\{(E\cap F)\cup(\bar{E}\cap F)\}}{P(F)} \quad [\because E\cap F \text{ and } \bar{E}\cap F \text{ are disjoint}]$$

$$=\frac{P\{(E\cup\bar{E})\cap F\}}{P(F)}=\frac{P(F)}{P(F)}=1$$

2. ***(a)*** A is independent to itself if and only if

$$P(A\cap A)=P(A)\cdot P(A) \Rightarrow P(A)=P(A)^2$$

$$\Rightarrow \quad P(A)[1-P(A)]=0 \quad \Rightarrow P(A)=0 \text{ or } P(A)=1$$

3. ***(a)*** We have,

$$P(A\cap B)=p$$

and $$P(A)+P(B)-2P(A\cap B)=q$$

$$\Rightarrow \quad P(A)+P(B)-2p=q$$

$$\Rightarrow \quad P(A)+P(B)=2p+q$$

$$\Rightarrow \quad 1-P(\bar{A})+1-P(\bar{B})=2p+q$$

$$\Rightarrow \quad P(\bar{A})+P(\bar{B})=2-2p-q$$

So, alternative (b) is correct.

Now, $$P\{(A\cap B)/(A\cup B)\}=\frac{P\{(A\cap B)\cap(A\cup B)\}}{P(A\cup B)}$$

$$=\frac{P(A\cap B)}{P(A\cup B)}=\frac{P(A\cap B)}{P(A)+P(B)-P(A\cap B)}$$

$$=\frac{p}{2p+q-p}=\frac{p}{p+q}$$

So, alternative (c) is correct.

Finally, $$P(\bar{A}\cap\bar{B})=P(\overline{A\cup B})=1-P(A\cup B)$$

$$=1-[P(A)+P(B)-P(A\cap B)]$$

$$=1-[2p+q-p]=1-p-q$$

So, alternative (d) is correct.

4. ***(d)*** Let $A_i\,(i=1,2)$ denotes the event that ith plane hits the target.

Clearly, A_1 and A_2 are independent events.

Required probability $=P(\bar{A}_1\cap A_2)=P(\bar{A}_1)P(A_2)$

$$=(1-0.3)\times 0.2=0.14$$

5. ***(c)*** Let E_1 denotes the event "a coin with two heads is selected" and E_2 denotes the event "a fair coin is selected". Let A be the event "the toss results in head". Then,

$$P(E_1)=\frac{1}{n+1},\ P(E_2)=\frac{n}{n+1},\ P\left(\frac{A}{E_1}\right)=1 \text{ and } P\left(\frac{A}{E_2}\right)=\frac{1}{2}$$

$$\therefore \quad P(A)=P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)$$

$$\Rightarrow \quad \frac{7}{12}=\frac{1}{n+1}\times 1+\frac{n}{n+1}\times\frac{1}{2} \qquad \left[\because P(A)=\frac{7}{12}\right]$$

$$\Rightarrow \quad 12+6n=7n+7 \Rightarrow n=5$$

6. ***(b)*** We have, $$P(u_i)\propto i; \qquad i=1,2,\ldots,n$$

$$\Rightarrow \quad P(u_i)=\lambda i; \qquad i=1,2,\ldots,n$$

But, $$\sum_{i=1}^{n}P(u_i)=1$$

$$\therefore \quad \sum_{i=1}^{n}\lambda i=1$$

$$\Rightarrow \quad \frac{\lambda n(n+1)}{2}=1 \Rightarrow \lambda=\frac{2}{n(n+1)}$$

Now, $$P(W)=\sum_{i=1}^{n}P(u_i\cap W)$$ [using total probability theorem]

$$\Rightarrow \quad P(W)=\sum_{i=1}^{n}P(u_i)P\left(\frac{W}{u_i}\right) \quad \Rightarrow \quad P(W)=\sum_{i=1}^{n}\lambda i\times\frac{i}{n+1}$$

$$\Rightarrow \quad P(W)=\frac{\lambda}{n+1}\sum_{i=1}^{n}i^2=\frac{\lambda}{n+1}\times\frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow \quad P(W)=\frac{2}{n(n+1)}\times\frac{n(2n+1)}{6} \Rightarrow P(W)=\frac{2n+1}{3(n+1)}$$

$$\therefore \quad \lim_{n\to\infty}P(W)=\lim_{n\to\infty}\frac{2n+1}{3(n+1)}=\frac{2}{3}$$

7. ***(c)*** Let A denotes the event that each American man is seated adjacent to his wife and B denotes the event that Indian man is seated adjacent to his wife. Then, required probability $=P(B/A)$

$$=\frac{\text{Number of ways in which Indian man sits adjacent to his wife when each man is seated adjacents to his wife}}{\text{Number of ways in which each American man is seated adjacent to his wife}}$$

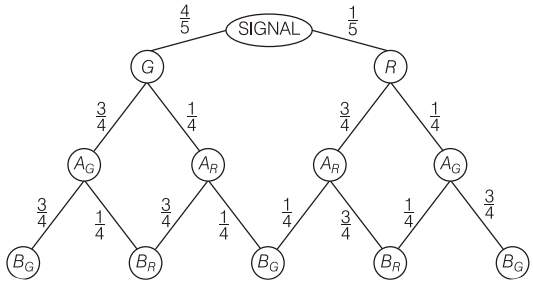
$$=\frac{(2!)^5\times(5-1)!}{(2!)^4\,(6-1)!}=\frac{2}{5}$$

8. ***(c)*** It is evident from the tree diagram that

$$P(B_G)=\frac{4}{5}\times\frac{3}{4}\times\frac{3}{4}+\frac{4}{5}\times\frac{1}{4}\times\frac{1}{4}+\frac{1}{5}\times\frac{3}{4}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{4}\times\frac{3}{4}=\frac{23}{40}$$

$$P\left(\frac{B_G}{G}\right)=\frac{3}{4}\times\frac{3}{4}+\frac{1}{4}\times\frac{1}{4}=\frac{5}{8}$$

$$\therefore \quad P(B_G\cap G)=P(G)P\left(\frac{B_G}{G}\right)=\frac{4}{5}\times\frac{5}{8}=\frac{1}{2}$$



$$\therefore \text{ Required probability } = P\left(\frac{G}{B_G}\right)=\frac{P(B_G\cap G)}{P(B_G)}=\frac{1/2}{23/40}=\frac{20}{23}$$

9. ***(a)*** We have, $$P\left(\frac{B}{A\cup\bar{B}}\right)=\frac{P[B\cap(A\cup\bar{B})]}{P(A\cup\bar{B})}$$

$$=\frac{P[(B\cap A)\cup(B\cap\bar{B})]}{P(A)+P(\bar{B})-P(A\cap\bar{B})}=\frac{P(A\cap B)}{P(A)+P(\bar{B})-P(A\cap\bar{B})}$$

Now, $$P(\bar{A}\cap\bar{B})=0.5$$

$$\therefore \quad 1-P(A\cup B)=0.5$$

$$\Rightarrow \quad P(A\cup B)=0.5$$

$$\Rightarrow \quad P(A)+P(B)-P(A\cap B)=0.5$$

$$\Rightarrow \quad P(A\cap B)=0.3+0.4-0.5=0.2$$

$$\therefore \quad P\left(\frac{B}{A\cup\bar{B}}\right)=\frac{0.2}{0.3+(1-0.4)-(0.3-0.2)}=\frac{0.2}{0.8}=\frac{1}{4}=0.25$$

10. ***(b)*** Let the events be

A : Exactly one child ; B : Exactly two children

C : Exactly three children

E : Couple has exactly 4 grandchildren

$\therefore\ P(A)=1/4;\quad P(B)=1/2;\quad P(C)=1/4$

$\therefore\ P(E)=P(A)\cdot P(E/A)+P(B)\cdot P(E/B)+P(C)\cdot P(E/C)$

Now, when couple having one child A, then A must have four children, which is not possible.

When couple has two children, either each child has two children or one has three children and other has one child.

When couple has three children, any two of children have one child each and remaining has two children.

$$\therefore P(E)=\frac{1}{4}\times 0+\frac{1}{2}\left[\underbrace{\left(\frac{1}{2}\right)^2}_{(1,\,2)}+\underbrace{\frac{1}{4}\times\frac{1}{4}\times 2}_{(1,\,3)\text{ or }(3,\,1)}\right]+\frac{1}{4}\left[\underbrace{3\left(\frac{1}{4}\times\frac{1}{4}\times\frac{1}{2}\right)}_{(1,\,1,\,2)}\right]$$

$$=\frac{1}{8}+\frac{1}{16}+\frac{3}{128}=\frac{27}{128}$$

11. ***(c)*** Let E_1 denotes the event that one of the first n urns is chosen and E_2 denotes the event that $(n+1)$th urn is selected. A denotes that event that two balls drawn are black. Then,

$$P(E_1)=\frac{n}{(n+1)},\ P(E_2)=\frac{1}{(n+1)}$$

$$P\left(\frac{A}{E_1}\right)=\frac{{}^6C_2}{{}^{10}C_2}=\frac{1}{3}\ \text{ and }\ P\left(\frac{A}{E_2}\right)=\frac{{}^5C_2}{{}^{10}C_2}=\frac{2}{9}$$

Using Baye's theorem, the required probability is

$$P\left(\frac{E_2}{A}\right)=\frac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$\Rightarrow\quad \frac{1}{16}=\frac{\left(\frac{1}{n+1}\right)\frac{2}{9}}{\left(\frac{n}{n+1}\right)\left(\frac{1}{3}\right)+\left(\frac{1}{n+1}\right)\frac{2}{9}}\Rightarrow\frac{1}{16}=\frac{2}{3n+2}$$

$$\Rightarrow\quad n=10$$

12. ***(d)*** Let R be the event that a red face appears in each of the first n throws.

E_1 : Die A is used when head has already fallen

E_2 : Die B is used when tail has already fallen

$$\therefore\quad P\left(\frac{R}{E_1}\right)=\left(\frac{2}{3}\right)^n\text{ and }P\left(\frac{R}{E_2}\right)=\left(\frac{1}{3}\right)^n$$

As per the given condition, $\dfrac{P(E_1)P\left(\frac{R}{E_1}\right)}{P(E_1)P\left(\frac{R}{E_1}\right)+P(E_2)P\left(\frac{R}{E_2}\right)}=\frac{32}{33}$

$$\Rightarrow\quad \frac{\frac{1}{2}\left(\frac{2}{3}\right)^n}{\frac{1}{2}\left(\frac{2}{3}\right)^n+\frac{1}{2}\left(\frac{1}{3}\right)^n}=\frac{32}{33}\Rightarrow\frac{2^n}{2^n+1}=\frac{32}{33}\Rightarrow n=5$$

13. ***(d)*** Let A : Exactly one ace; B Both aces ; E : $A\cup B$

$$P\left(\frac{B}{A\cup B}\right)=\frac{{}^4C_2}{{}^4C_1\ {}^{12}C_1+{}^4C_2}=\frac{6}{54}=\frac{1}{9}$$

14. ***(a)*** We have, $\quad P(A)=P(B)=P(C)$

and $\quad P(A)+P(B)+P(C)=1$

$\therefore\quad P(A)=P(B)=P(C)=1/3$

Also, $\quad P(X)=\frac{5}{12},\ P\left(\frac{X}{A}\right)=\frac{3}{8},\ P\left(\frac{X}{B}\right)=\frac{1}{4}$

We have, $\quad P(X)=P(A)P\left(\frac{X}{A}\right)+P(B)P\left(\frac{X}{B}\right)+P(C)P\left(\frac{X}{C}\right)$

$$\therefore\quad \frac{5}{12}=\frac{1}{3}\left\{\frac{3}{8}+\frac{1}{4}+P\left(\frac{X}{C}\right)\right\}\Rightarrow P\left(\frac{X}{C}\right)=\frac{5}{8}$$

15. ***(a)*** E_1 : Event that the dot is removed from an odd face;

$$\therefore\quad P(E_1)=\frac{1+3+5}{21}=\frac{9}{21}$$

E_2 : Dot is removed from the even face

$$\therefore\quad P(E_2)=\frac{2+4+6}{21}=\frac{12}{21}$$

E : Odd number appears when die is thrown

Also, when dot is removed from an odd numbered face, there are exactly 2 faces with odd number of dots and when dot is removed from an even numbered face, there are exactly 4 faces with odd number of dots.

$\therefore$ Required probability

$$P(E)=P(E\cap E_1)+P(E\cap E_2)$$
$$=P(E_1)\cdot P(E/E_1)+P(E_2)P(E/E_2)$$
$$=\frac{9}{21}\times\frac{2}{6}+\frac{12}{21}\times\frac{4}{6}=\frac{11}{21}$$

16. ***(a)*** We know that, $P(A\cup\overline{B})=1-P(\overline{A\cup\overline{B}})$

$$=1-P(\overline{A}\cap B)=1-P(\overline{A})\cdot P(B)$$

$$\Rightarrow\quad 0.9=1-0.6\times P(B)\Rightarrow P(B)=1/6$$

17. ***(b)*** We define the following events

A_1 : Selecting a pair of consecutive letters from the word LONDON

A_2 : Selecting a pair of consecutive letters from the word CLIFTON

E : Selecting a pair of letter 'ON'

Then, $P(A_1\cap E)=\frac{2}{5}$ as there are 5 pairs of consecutive letters out of which 2 are ON and $P(A_2\cap E)=\frac{1}{6}$ as there are 6 pairs of consecutive letters of which 1 is ON. Therefore, the required probability is

$$P\left(\frac{A_1}{E}\right)=\frac{P(A_1\cap E)}{P(A_1\cap E)+P(A_2\cap E)}=\frac{2/5}{2/5+1/6}=\frac{12}{17}$$

18. ***(d)*** We have, P (Exactly one of A, B occurs)

$$=P[(A\cap\overline{B})\cup(\overline{A}\cap B)]=P(A\cap\overline{B})+P(\overline{A}\cap B)$$
$$=P(A)-P(A\cap B)+P(B)-P(A\cap B)$$
$$=P(A)+P(B)-2P(A\cap B)=P(A\cup B)-P(A\cap B)$$

Also, P(Exactly one of A, B occurs)

$$=P(A\cup B)-P(A\cap B)=\{1-P(\overline{A\cup B})\}-\{1-P(\overline{A\cap B})\}$$
$$=[1-P(\overline{A}\cap\overline{B})]-[1-P(\overline{A}\cup\overline{B})]=P(\overline{A}\cup\overline{B})-P(\overline{A}\cap\overline{B})$$
$$=P(\overline{A})+P(\overline{B})-2P(\overline{A}\cap\overline{B})$$

19. ***(d)*** The minimum of two numbers is less than 4 means atleast one of them is less than 4.

Let A_i denotes the event that the number selected in ith draw is less than 4, where $i=1,2$. Then,

Required probability $=P(A_1\cup A_2)=1-P(\overline{A}_1\cap\overline{A}_2)$

$$=1-P(\overline{A}_1)\,P\left(\frac{\overline{A}_2}{\overline{A}_1}\right)=1-\frac{3}{6}\times\frac{2}{5}=\frac{4}{5}$$

20. ***(a)*** We have, required probability $=1-$ probability that none of the selected horse is a winning horse

$$=1-\left(\frac{4}{5}\times\frac{3}{4}\right)=1-\frac{3}{5}=\frac{2}{5}$$

21. ***(b)*** Required probability $= P(A_1' \cap A_2' \cap \ldots \cap A_n')$

$= P(A_1')\,P(A_2')\ldots P(A_n')$ $[\because A_1, A_2, \ldots, A_n$ are independent$]$

$= [1-P(A_1)][1-P(A_2)]\ldots[1-P(A_n)]$

$= \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n+1}\right)$

$= \frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\ldots\times\frac{n-1}{n}\times\frac{n}{n+1} = \frac{1}{n+1}$

22. ***(a)*** We know that, $P\left(\frac{C}{D}\right) = \frac{P(C\cap D)}{P(D)} = \frac{P(C)}{P(D)}$ $[\because C \subseteq D]$

23. ***(b)*** $(Y=0)$ is {00, 01, ..., 09, 10, 20, ... , 90}

Also, $(X=9)\cap(Y=0) = \{09, 90\}$.

We have $P(Y=0) = 19/100$

and $P[(X=9)\cap(Y=0)] = 2/100$

$\therefore \quad P(X=9/Y=0) = \frac{P[(X=9)\cap(Y=0)]}{P(Y=0)} = \frac{2}{19}$

24. ***(c)*** Let A, B, C and S denote the following events

A : A solves the problem correctly

B : B solves the problem correctly

C : A and B make common errors

S : A and B reach the same answer

We have, $S = (A\cap B)\cup(A'\cap B'\cap C)$

$\Rightarrow \quad P(S) = P(A\cap B) + P(A'\cap B'\cap C)$

$= \frac{1}{4}\times\frac{1}{5} + \frac{3}{4}\times\frac{4}{5}\times\frac{1}{40} = \frac{13}{200}$

Now, P(answer is correct/s)

$= \frac{P(A\cap B\cap S)}{P(S)} = \frac{P(A\cap B)}{P(S)} = \frac{1/4\times1/5}{13/200} = \frac{10}{13}$

25. ***(a)*** We have, $P(A/B) = P(A/B')$

$\Rightarrow \quad \frac{P(A\cap B)}{P(B)} = \frac{P(A\cap B')}{P(B')}$

$\Rightarrow \quad \frac{P(A\cap B)}{P(B)} = \frac{P(A)-P(A\cap B)}{1-P(B)}$

$\Rightarrow P(A\cap B)[1-P(B)] = P(A)\cdot P(B) - P(B)\cdot P(A\cap B)$

$\Rightarrow \quad P(A\cap B) = P(A)\cdot P(B)$

Hence, A and B are independent.

26. ***(b)*** We have, $P(A) = P(A/B) = 1/4$

Shows that A and B are independent events.

$P(B) = P(B/A) = 1/2$

Now, $P(A/B) = \frac{1}{4} \Rightarrow P(\bar{A}/B) = 1-\frac{1}{4} = \frac{3}{4}$

27. ***(d)*** We have,

$P(X_1) = \frac{1}{2}, P(X_2) = \frac{1}{4}, P(X_3) = \frac{1}{4}$

$P(X) = P[(X_1\cap X_2\cap X_3^c)\cup(X_1\cap X_2^c\cap X_3)\cap(X_1^c\cap X_2\cap X_3)$

$\cup(X_1\cap X_2\cap X_3)]$

$= P(X_1\cap X_2\cap X_3^c) + P(X_1\cap X_2^c\cap X_3) + P(X_1^c\cap X_2\cap X_3)$

$+ P(X_1\cap X_2\cap X_3)$

$= P(X_1)P(X_2)P(X_3^c) + P(X_1)P(X_2^c)P(X_3)$

$+ P(X_1^c)P(X_2)P(X_3) + P(X_1)P(X_2)P(X_3)$

$= \frac{1}{2}\times\frac{1}{4}\times\frac{3}{4} + \frac{1}{2}\times\frac{3}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{1}{4} = \frac{1}{4}$

Now, $P(X_1^c/X) = \frac{P(X_1^c\cap X)}{P(X)} = \frac{P(X_1^{\,c}\cap X_2\cap X_3)}{P(X)} = \frac{\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4}}{1/4} = \frac{1}{8}$

P (Exactly two engines of the ship are functioning/X)

$= \frac{P\{(X_1^c\cap X_2\cap X_3)\cap(X_1\cap X_2^c\cap X_3)\cap(X_1\cap X_2\cap X_3^c)\}}{P(X)}$

$= \frac{P(X_1^c\cap X_2\cap X_3) + P(X_1\cap X_2^c\cap X_3) + P(X_1\cap X_2\cap X_3^c)}{P(X)}$

$= \frac{\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{3}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{3}{4}}{\frac{1}{4}} = \frac{7}{8}$

$P\left(\frac{X}{X_2}\right) = \frac{P(X\cap X_2)}{P(X_2)}$

$= \frac{P\{(X_1^c\cap X_2\cap X_3)\cap(X_1\cap X_2\cap X_3^c)\cap(X_1\cap X_2\cap X_3)\}}{P(X_2)}$

$= \frac{P(X_1^c\cap X_2\cap X_3) + P(X_1\cap X_2\cap X_3^{\,c}) + P(X_1\cap X_2\cap X_3)}{P(X_2)}$

$= \frac{\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{3}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{1}{4}}{1/4} = \frac{5/32}{1/4} = \frac{5}{8}$

$P(X/X_1) = \frac{P(X\cap X_1)}{P(X_1)}$

$= \frac{P\{(X_1\cap X_2\cap X_3)\cap(X_1\cap X_2\cap X_3^c)\cap(X_1\cap X_2^c\cap X_3)\}}{P(X_1)}$

$= \frac{P(X_1\cap X_2\cap X_3) + P(X_1\cap X_2\cap X_3^c) + P(X_1\cap X_2^c\cap X_3)}{P(X_1)}$

$= \frac{\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{3}{4} + \frac{1}{2}\times\frac{3}{4}\times\frac{1}{4}}{1/2} = \frac{7}{16}$

28. ***(d)*** We know that,

$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)+P(B)-P(A\cup B)}{P(B)}$

Now, $P(A\cup B) < 1$

$\Rightarrow \quad -P(A\cup B) > -1$

$\Rightarrow \quad P(A)+P(B)-P(A\cup B) > P(A)+P(B)-1$

$\Rightarrow \quad \frac{P(A)+P(B)-P(A\cup B)}{P(B)} > \frac{P(A)+P(B)-1}{P(B)}$

$\Rightarrow \quad P(A/B) > \frac{P(A)+P(B)-1}{P(B)}$

Thus, choice (a) is correct.

We know that, $P(A\cap\bar{B}) = P(A) - P(A\cap B)$

So, option (b) is correct.

For independent events A and B, we have

$P(A\cup B) = 1 - P(\bar{A})\,P(\bar{B})$

So, choice (c) is also correct. The choice (d) is incorrect as it is true for independent events only.

29. ***(c)*** Consider the following events

A = Selecting a month of the year

B = 13th day of a month is Friday

We have, $P(A) = 1/12$ and $P(B/A) = 1/7$

$\therefore$ Required probability $= P(A\cap B) = P(A)\,P\left(\frac{B}{A}\right) = \frac{1}{12}\times\frac{1}{7} = \frac{1}{84}$

30. ***(d)*** Let A_i denotes the event that the student is present in the test held on ith day, where $i = 1, 2$

$\therefore$ Required probability $= 1 - P$ (Student does not attend any test)

$= 1 - P(\bar{A}_1\cap\bar{A}_2)$

$= 1 - P(\bar{A}_1)\,P(\bar{A}_2) = 1-(4/5)\times(4/5) = 9/25$

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❶ INTEGRAL POWERS OF IOTA (i)

Euler was the first one who introduces the symbol i (iota) for the square root of -1 with the property $i^2 = -1$. He also called this symbol as the imaginary unit.

Ⓐ POSITIVE INTEGRAL POWERS OF i

We have, $i = \sqrt{-1}$

$$i^2 = -1,\ i^3 = i^2 \cdot i = -i,$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

The value of i^n for $n > 4$ is i^r, where r is the remainder when n is divided by 4.

Ⓑ NEGATIVE INTEGRAL POWERS OF i

According to the properties of indices, we have,

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i,\ i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$$

$$i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i,\ i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

If $n > 4$, then $i^{-r} = \frac{1}{i^r}$, where r is the remainder when n is divided by 4.

Note $(i)^0$ is defined as 1.

❷ IMAGINARY QUANTITIES

The square root of a negative real number is called an imaginary quantity or an imaginary number.

❸ COMPLEX NUMBERS

If a, b are two real numbers, then a number of the form $a + ib$ is called a complex number.

If $z = a + ib$ is a complex number, then 'a' is called the real part of z and 'b' is known as the imaginary part of z. The real part of z is denoted by $\mathrm{Re}(z)$ and the imaginary part by $\mathrm{Im}(z)$.

A complex number z is purely real, if its imaginary part is zero, i.e. $\mathrm{Im}(z) = 0$ and purely imaginary, if its real part is zero, i.e. $\mathrm{Re}(z) = 0$.

❹ EQUALITY OF COMPLEX NUMBERS

The complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal, if $a_1 = a_2$ and $b_1 = b_2$, i.e. $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

❺ FUNDAMENTAL OPERATIONS ON COMPLEX NUMBERS

Following are some operations which can be apply on complex numbers

Ⓐ ADDITION

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then, their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$.

Clearly, $\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2)$

and $\mathrm{Im}(z_1 + z_2) = \mathrm{Im}(z_1) + \mathrm{Im}(z_2)$

PROPERTIES OF ADDITION

(i) **Addition is commutative** For any two complex numbers z_1 and z_2, we have $z_1 + z_2 = z_2 + z_1$

(ii) **Addition is associative** For any three complex numbers z_1, z_2, z_3, we have

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

(iii) **Existence of additive identity** The complex number $0 = 0 + i0$ is the identity element for addition, i.e. $z + 0 = z = 0 + z,\quad z \in C$

(iv) **Existence of additive inverse** For every complex number z there exists $(-z)$ such that $z + (-z) = 0 = (-z) + z$. The complex number $-z$ is called the additive inverse of z.

Ⓑ SUBTRACTION

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then, the subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and $-z_2$.

Thus, $z_1 - z_2 = z_1 + (-z_2)$

$$= (a_1 + ib_1) + (-a_2 - ib_2)$$

$$= (a_1 - a_2) + i(b_1 - b_2)$$

Ⓒ MULTIPLICATION

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then, the multiplication of z_1 with z_2 is denoted by $z_1 z_2$ and is defined as the complex number

$$(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

PROPERTIES OF MULTIPLICATION

(i) **Multiplication is commutative**
For any two complex numbers z_1 and z_2, we have

$$z_1 z_2 = z_2 z_1$$

(ii) **Multiplication is associative**
For any three complex numbers z_1, z_2, z_3, we have

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

(iii) **Existence of identity element for multiplication**
The complex number $1 = 1 + i0$ is the identity element for multiplication, i.e. for every complex number z, we have $z \times 1 = z = 1 \times z$

(iv) **Existence of multiplicative inverse**
Corresponding to every non-zero complex number $z = a + ib$ there exists a complex number $z_1 = x + iy$ such that $z \times z_1 = 1 = z_1 \times z$
The complex number z_1 is called the multiplicative inverse or reciprocal of z and is given by

$$z_1 = \frac{a}{a^2 + b^2} + \frac{i(-b)}{a^2 + b^2}$$

(v) **Multiplication of complex number is distributive over addition of complex number**
For any three complex numbers z_1, z_2, z_3, we have

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

Ⓓ DIVISION

The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.

Thus, $\frac{z_1}{z_2} = z_1 \times z_2^{-1} = z_1 \left(\frac{1}{z_2}\right)$

CUBE ROOT

If $z = (1)^{1/3}$

$\Rightarrow \alpha = e^{i 2\pi/3}$

So, cube roots of unity are $1, \alpha, \alpha^2$,

i.e. $1, e^{i 2\pi/3}$ and $e^{i 4\pi/3}$

We denote $e^{i 2\pi/3}$ by ω,

i.e. $\omega = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Thus, cube roots of unity are $1, \omega, \omega^2$, where

$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Properties of Cube Roots of Unity

(i) Cube roots of unity are $1, \omega, \omega^2$.

(ii) $\arg(\omega) = \frac{2\pi}{3}$ and $\arg(\omega^2) = \frac{4\pi}{3}$

(iii) $z^3 - 1 = (z - 1)(z - \omega)(z - \omega^2)$

(iv) ω and ω^2 are roots of the equation $z^2 + z + 1 = 0$

Ⓔ CONJUGATE

Let $z = a + ib$ be a complex number. Then, the conjugate of z is denoted by $\bar{z}$ and is equal to $a - ib$.

Thus, $z = a + ib$

$\Rightarrow \bar{z} = a - ib$

PROPERTIES OF CONJUGATE

If z, z_1, z_2 are complex numbers, then

(i) $\overline{(\bar{z})} = z$ (ii) $z + \bar{z} = 2\,\mathrm{Re}(z)$

(iii) $z - \bar{z} = 2i\,\mathrm{Im}(z)$

(iv) $z = \bar{z} \Leftrightarrow z$ is purely real

(v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

(vi) $z\bar{z} = \{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2$

(vii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (viii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(ix) $\overline{z_1 z_2} = \bar{z}_1 \times \bar{z}_2$

(x) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2},\ z_2 \neq 0$

❻ MODULUS OF A COMPLEX NUMBER

The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2}$$

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Numbers

'S OF UNITY

(v) Cube roots of unity lie on the unit circle $|z|=1$ and divide its circumference into three equal parts.

(vi) $1+\omega^n+\omega^{2n}=\begin{cases}0, & \text{if } n \text{ is not a multiple of 3}\\ 3, & \text{if } n \text{ is a multiple of 3}\end{cases}$

(vii) Cube roots of (-1) are $-1, -\omega, -\omega^2$.

(viii) $z^3+1=(z+1)(z+\omega)(z+\omega^2)$

(ix) $-\omega$ and $-\omega^2$ are roots of $z^2-z+1=0$

(x) $a^3+b^3=(a+b)(a+b\omega)(a+b\omega^2)$

$a^3-b^3=(a-b)(a-b\omega)(a-b\omega^2)$

$a^2+b^2+c^2-ab-bc-ca$
$=(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

$a^3+b^3+c^3-3abc=(a+b+c)$
$(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

(xi) $1+\omega+\omega^2=0$

(xii) $\omega^{3n}=1$

PROPERTIES OF MODULUS

If $z, z_1, z_2 \in C$, then

(i) $|z|=0 \Leftrightarrow z=0$,
i.e. $\text{Re}(z)=\text{Im}(z)=0$

(ii) $|z|=|\bar{z}|=|-z|$

(iii) $-|z| \le \text{Re}(z) \le |z|$; $-|z| \le \text{Im}(z) \le |z|$

(iv) $z\cdot\bar{z}=|z|^2$

(v) $|z_1 z_2|=|z_1||z_2|$

(vi) $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}, z_2 \neq 0$

(vii) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\,\text{Re}(z_1\bar{z}_2)$

(viii) $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2\,\text{Re}(z_1\bar{z}_2)$

(ix) $|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$

(x) $|az_1-bz_2|^2+|bz_1+az_2|^2$
$=(a^2+b^2)(|z_1|^2+|z_2|^2)$

7 RECIPROCAL OF A COMPLEX NUMBER

Let z be a complex number. Then,

$$\frac{1}{z}=\frac{\bar{z}}{z\bar{z}}=\frac{\bar{z}}{|z|^2}$$

8 SQUARE ROOTS OF A COMPLEX NUMBER

The square root of a complex number z is given by

$$\sqrt{z}=\pm\left[\sqrt{\frac{1}{2}\{|z|+\text{Re}(z)\}}+i\sqrt{\frac{1}{2}\{|z|-\text{Re}(z)\}}\right]$$

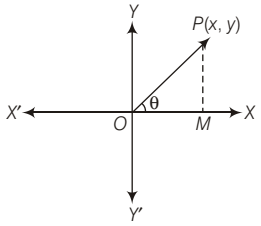
if $\text{Im}(z)>0$ and

$$\sqrt{z}=\pm\left[\sqrt{\frac{1}{2}\{|z|+\text{Re}(z)\}}-i\sqrt{\frac{1}{2}\{|z|-\text{Re}(z)\}}\right]$$

if $\text{Im}(z)<0$

9 ARGUMENT OF A COMPLEX NUMBER

The angle θ which $|z|$ vector makes with positive direction of X-axis in anti-clockwise sense is called the argument or amplitude of z and is denoted by $\arg(z)$ or $\text{amp}(z)$.



$$\therefore \quad \tan\theta=\frac{PM}{OM}=\frac{y}{x}=\frac{\text{Im}(z)}{\text{Re}(z)}$$

$$\Rightarrow \quad \theta=\tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$$

This angle θ has infinitely many values differing by multiples of 2π. The unique value of θ such that $-\pi<\theta\le\pi$ is called the principal value of the amplitude or principal argument.

ALGORITHM

Step I Find the value of $\tan^{-1}\left|\frac{y}{x}\right|$ lying between 0 and $\pi/2$. Let it be α.

Step II Determine in which quadrant the point $P(x, y)$ belongs. If $P(x, y)$ belongs to the first quadrant, then $\arg(z)=\alpha$.

If $P(x, y)$ belongs to the second quadrant, then $\arg(z)=\pi-\alpha$.

If $P(x, y)$ lies in the third quadrant, then $\arg(z)=\pi+\alpha$.

If $P(x, y)$ belongs to the fourth quadrant, then $\arg(z)=-\alpha$, or $2\pi-\alpha$.

10 SOME IMPORTANT RESULTS ON MODULUS AND ARGUMENT

Let z, z_1, z_2 be complex numbers with arguments θ_1 and θ_2, respectively.

Then,

(i) $\arg(\bar{z})=-\arg(z)$

(ii) $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$

(iii) $\arg(z_1\bar{z}_2)=\arg(z_1)-\arg(z_2)$

(iv) $\arg(z_1/z_2)=\arg(z_1)-\arg(z_2)$

(v) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$

(vi) $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$

(vii) $|z_1+z_2|=|z_1-z_2| \Leftrightarrow \arg(z_1)-\arg(z_2)=\pi/2$

(viii) $|z_1+z_2|=|z_1|+|z_2| \Leftrightarrow \arg(z_1)=\arg(z_2)$

11 DE-MOIVRE'S THEOREM

If $n \in z$, then

(i) $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

(ii) $(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$

(iii) $\frac{1}{\cos\theta+i\sin\theta}=(\cos\theta+i\sin\theta)^{-1}$
$=\cos\theta-i\sin\theta$

(iv) $(\sin\theta+i\cos\theta)^n \neq \sin n\theta+i\cos n\theta$

12 ROOTS OF A COMPLEX NUMBER

Let $z=(a+ib)$ be a complex number and r be its modulus and θ be its argument, then

$$z^{1/n}=r^{1/n}\left\{\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right\}$$

13 nTH ROOTS OF UNITY

Let $z=1^{1/n}$

$\Rightarrow z=(\cos 0°+i\sin 0°)^{1/n}$

$\Rightarrow z=(\cos 2r\pi+i\sin 2r\pi)^{1/n}$, where $r \in z$

$\Rightarrow z=\cos\frac{2r\pi}{n}+i\sin\frac{2r\pi}{n}$,
where $r=0, 1, \ldots, (n-1)$

$\Rightarrow z=e^{i2r\pi/n}, r=0, 1, 2, \ldots, (n-1)$
$[\because e^{i\theta}=\cos\theta+i\sin\theta]$

$\Rightarrow z=e^{\left(\frac{i2\pi}{n}\right)^r}, r=0, 1, 2, \ldots, (n-1)$

$\Rightarrow z=\alpha^r$, where $\alpha=e^{i2\pi/n}$

Thus, nth roots of unity are

$$1(=\alpha^0), \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1},$$

where $\alpha=e^{i2\pi/n}=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$

PROPERTIES OF nTH ROOTS OF UNITY

(i) nth roots of unity form a GP with common ratio $e^{i2\pi/n}$.

(ii) Sum of nth roots of unity is always zero.

(iii) Sum of pth powers of nth roots of unity is zero, if p is not a multiple of n.

(iv) Sum of pth powers of nth roots of unity is n, if p is a multiple of n.

(v) Product of nth roots of unity is $(-1)^{n-1}$.

(vi) nth roots of unity lie on the unit circle $|z|=1$ and divide its circumference into n equal parts.

Love Agarwal

PROPERTIES OF DEFINITE INTEGRAL

Coverage of special topics for JEE Advanced

Properties of definite integral is very helpful in finding the value of definite integrals. In this article we will discuss about fundamental properties of definite integrals and also we will learn how to use them.

Definite Integral

Let $\phi(x)$ be the primitive or anti-derivative of a function $f(x)$ defined on $[a, b]$, i.e. $\frac{d}{dx}\{\phi(x)\} = f(x)$. Then, the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)\,dx$ and is defined as $[\phi(b) - \phi(a)]$.

i.e. $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$

The numbers a and b are called the limits of integration, 'a' is called lower limit and 'b' the upper limit. The interval $[a, b]$ is called the interval of integration.

Properties of Definite Integral

Following are the properties of definite integral.

Property I

Changing dummy variable $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$, i.e. the value of the definite integral does not change with the change of argument (variable of integration) provided the limits of integration remain the same.

Property II

Interchanging limits $\int_a^b f(x)dx = -\int_b^a f(x)\,dx$, i.e. the sign of the definite integral is changed when the order of the limits changes.

Property III

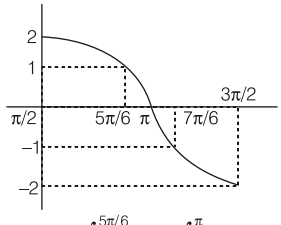
Splitting limits $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where c may lie inside or outside the interval $[a, b]$.

Example 1 If for a real number y, $[y]$ is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2}[2\sin x]dx$ is [IIT JEE 2002]

(a) $-\pi$
(b) 0
(c) $-\frac{\pi}{2}$
(d) $\frac{\pi}{2}$

Sol. **(c)** The graph of $y = 2\sin x$ for $\pi/2 \le x \le 3\pi/2$ is given in figure. From the graph, it is clear that

$$[2\sin x] = \begin{cases} 2, & \text{if } x = \frac{\pi}{2} \\ 1, & \text{if } \frac{\pi}{2} < x \le \frac{5\pi}{6} \\ 0, & \text{if } \frac{5\pi}{6} < x \le \pi \\ -1, & \text{if } \pi < x \le \frac{7\pi}{6} \\ -2, & \text{if } \frac{7\pi}{6} < x \le \frac{3\pi}{2} \end{cases}$$



$$\therefore \int_{\pi/2}^{3\pi/2}[2\sin x]\,dx = \int_{\pi/2}^{5\pi/6} dx + \int_{5\pi/6}^{\pi} 0\,dx + \int_{\pi}^{7\pi/6}(-1)\,dx + \int_{7\pi/6}^{3\pi/2}(-2)\,dx$$

$$= [x]_{\pi/2}^{5\pi/6} + [-x]_{\pi}^{7\pi/6} + [-2x]_{7\pi/6}^{3\pi/2}$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{2}\right) + \left(-\frac{7\pi}{6} + \pi\right) + \left(-2\cdot\frac{3\pi}{2} + 2\cdot\frac{7\pi}{6}\right)$$

$$= -\frac{\pi}{2}$$

Example 2 The value of the integral $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$ is

[IIT JEE 2000]

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) 3 (d) 5

Sol. **(b)** $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx = \int_{e^{-1}}^{1} \left|\frac{\log_e x}{x}\right| dx + \int_{1}^{e^2} \left|\frac{\log_e x}{x}\right| dx$

$= -\int_{e^{-1}}^{1} \frac{\log_e x}{x} dx + \int_{1}^{e^2} \frac{\log_e x}{x} dx$

$= -\frac{1}{2}[(\log_e x)^2]_{e^{-1}}^{1} + \frac{1}{2}[(\log_e x)^2]_{1}^{e^2}$

$= -\frac{1}{2}\{0 - (-1)^2\} + \frac{1}{2}\{2^2 - 0\} = \frac{5}{2}$

Example 3 The integral $\int_0^{\pi} \sqrt{1 + 4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\, dx$ is equal to

[IIT JEE 2014]

(a) $\pi - 4$ (b) $\frac{2\pi}{3} - 4 - 4\sqrt{3}$

(c) $4\sqrt{3} - 4$ (d) $4\sqrt{3} - 4 - \frac{\pi}{3}$

Sol. **(d)** $\int_0^{\pi} \sqrt{1 + 4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\, dx$

$= \int_0^{\pi} \sqrt{\left(1 - 2\sin\frac{x}{2}\right)^2}\, dx = \int_0^{\pi} \left|1 - 2\sin\frac{x}{2}\right| dx$

$= \int_0^{\pi/3} \left(1 - 2\sin\frac{x}{2}\right) dx - \int_{\pi/3}^{\pi} \left(1 - 2\sin\frac{x}{2}\right) dx$

$= \left[x + 4\cos\frac{x}{2}\right]_0^{\pi/3} - \left[x + 4\cos\frac{x}{2}\right]_{\pi/3}^{\pi}$

$= 4\sqrt{3} - 4 - \frac{\pi}{3}$

Property IV

$\int_a^b f(x)\,dx = \int_a^b f(a + b - x)dx$ and $\int_0^a f(x)\,dx = \int_0^a f(a - x)dx$

Example 4 Let f be a positive function.

Let $I_1 = \int_{1-k}^{k} xf[x(1-x)]\,dx$ and $I_2 = \int_{1-k}^{k} f[x(1-x)]\,dx$, where $2k - 1 > 0$. Then, $\frac{I_1}{I_2}$ is

[IIT JEE 2003]

(a) 2 (b) k (c) $\frac{1}{2}$ (d) 1

Sol. **(c)** Given, $I_1 = \int_{1-k}^{k} xf[x(1-x)]\,dx$

$\Rightarrow \quad I_1 = \int_{1-k}^{k} (1-x)\, f[(1-x)x]\,dx$

$\left[\because \int_a^b f(x)dx = \int_a^b f(a + b - x)dx\right]$

$\Rightarrow \quad I_1 = I_2 - I_1$

$\Rightarrow \quad 2I_1 = I_2 \Rightarrow \frac{I_1}{I_2} = \frac{1}{2}$

Example 5 $\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ is equal to

[IIT JEE 2005]

(a) 2 (b) -2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

Sol. **(a)** Let $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ ...(i)

$= \int_{\pi/4}^{3\pi/4} \frac{1}{1 + \cos(\pi - x)} dx$

$I = \int_{\pi/4}^{3\pi/4} \frac{1}{1 - \cos x} dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1 + \cos x} + \frac{1}{1 - \cos x}\right) dx$

$2I = \int_{\pi/4}^{3\pi/4} \frac{2}{1 - \cos^2 x} dx$

$\Rightarrow \quad I = \int_{\pi/4}^{3\pi/4} \text{cosec}^2 x\, dx = [-\cot x]_{\pi/4}^{3\pi/4}$

$= \left[-\cot\frac{3\pi}{4} + \cot\frac{\pi}{4}\right] = -(-1) + 1 = 2$

Example 6 The value of $\int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\log 6 - x^2)} dx$ is

[IIT JEE 2011]

(a) $\frac{1}{4}\log\left(\frac{3}{2}\right)$ (b) $\frac{1}{2}\log\left(\frac{3}{2}\right)$ (c) $\log\left(\frac{3}{2}\right)$ (d) $\frac{1}{6}\log\left(\frac{3}{2}\right)$

Sol. **(a)** Put $x^2 = t \Rightarrow x\,dx = \frac{dt}{2}$

$\therefore \quad I = \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin t\, dt}{\sin t + \sin(\log 6 - t)}$ ...(i)

$\Rightarrow \quad I = \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 2 + \log 3 - t)}{\sin(\log 2 + \log 3 - t) + \sin[\log 6 - (\log 2 + \log 3 - t)]} dt$

$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t} dt$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2I = \frac{1}{2}\int_{\log 2}^{\log 3} 1\, dt \quad \Rightarrow 2I = \frac{1}{2}[t]_{\log 2}^{\log 3}$

$\Rightarrow \quad 2I = \frac{1}{2}(\log 3 - \log 2) \Rightarrow I = \frac{1}{4}\log\left(\frac{3}{2}\right)$

Example 7 **Statement I** The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$ is equal to $\frac{\pi}{6}$.

Statement II $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

[IIT JEE 2013]

(a) Statement I is true, Statement II is true; Statement II is correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

Sol. ***(d)*** Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ ...(i)

$$\therefore \quad I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\,dx \qquad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} 1\,dx = [x]_{\pi/6}^{\pi/3}$$

$$\Rightarrow \quad 2I = \frac{\pi}{3} - \frac{\pi}{6} \Rightarrow 2I = \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}$$

Statement I is false.

But $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$ is a true statement.

Example 8 If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}\,dx$, $n = 0, 1, 2, \ldots$, then [IIT JEE 2009]

(a) $I_n = I_{n+2}$ (b) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(c) $\sum_{m=1}^{10} I_{2m} = 0$ (d) $I_n = I_{n+1}$

Sol. ***(a,b,c)*** Given, $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}\,dx$ ...(i)

Using $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

we get $I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(1+\pi^x)\sin x}\,dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x}dx = 2\int_0^{\pi} \frac{\sin nx}{\sin x}\,dx$$

$$\Rightarrow \quad I_n = \int_0^{\pi} \frac{\sin nx}{\sin x}\,dx$$

Now, $I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x}\,dx$

$$= \int_0^{\pi} \frac{2\cos(n+1)x \sin x}{\sin x}\,dx$$

$$= 2\int_0^{\pi} \cos(n+1)x\,dx$$

$$= 2\left[\frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi} = 0$$

$$\therefore \quad I_{n+2} = I_n \qquad \ldots(iii)$$

Since, $I_n = \int_0^{\pi} \frac{\sin nx}{\sin x}\,dx$

$$\Rightarrow \quad I_1 = \pi \quad \text{and} \quad I_2 = 0$$

From Eq. (iii), $I_1 = I_3 = I_5 = \ldots = \pi$

and $I_2 = I_4 = I_6 = \ldots = 0$

$$\Rightarrow \quad \sum_{m=1}^{10} I_{2m+1} = 10\pi \text{ and } \sum_{m=1}^{10} I_{2m} = 0$$

Example 9 Evaluate $\int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x}\,dx$. [IIT JEE 2002]

Sol. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x}\,dx$ ...(i)

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\cos^4\left(\frac{\pi}{2}-x\right)+\sin^4\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx - \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x}dx \quad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan x \cdot \sec^2 x}{1+\tan^4 x}\,dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\cdot\frac{1}{2}\int_0^{\pi/2} \frac{d(\tan^2 x)}{1+(\tan^2 x)^2}$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}[\tan^{-1}(\tan^2 x)]_0^{\pi/2}$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\cdot\frac{\pi}{2} \Rightarrow I = \frac{\pi^2}{16}$$

Example 10 Show that $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$ [IIT JEE 2004]

Sol. Let $I = \int_0^{\pi/2} f(\sin 2x)\sin x\,dx$ ...(i)

$$\therefore \quad I = \int_0^{\pi/2} f\left[\sin 2\left(\frac{\pi}{2}-x\right)\right]\sin\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_0^{\pi/2} f(\sin 2x)\cos x\,dx \qquad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x)\,dx$$

$$= 2\int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)\,dx$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin\left(x+\frac{\pi}{4}\right)dx$$

$$= 2\sqrt{2}\int_0^{\pi/4} f\left[\sin 2\left(\frac{\pi}{4}-x\right)\right]\sin\left(\frac{\pi}{4}-x+\frac{\pi}{4}\right)dx$$

$$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$$

$$\Rightarrow \quad I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$$ **Hence proved.**

Example 11 Prove that

$$\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = 2\int_0^1 \tan^{-1} x\,dx$$ [IIT JEE 2006]

Hence or otherwise, evaluate the integral

$$\int_0^1 \tan^{-1}(1-x+x^2)\,dx.$$

Sol. $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = \int_0^1 \tan^{-1}\left[\frac{1-x+x}{1-x(1-x)}\right]dx$

$= \int_0^1 [\tan^{-1}(1-x) + \tan^{-1} x]\,dx$

$= \int_0^1 \tan^{-1}[1-(1-x)]\,dx - \int_0^1 \tan^{-1} x\,dx$

$= 2\int_0^1 \tan^{-1} x\,dx$ ...(i)

Now, $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = \int_0^1\left[\frac{\pi}{2} - \cot^{-1}\left(\frac{1}{1-x+x^2}\right)\right]dx$

$= \frac{\pi}{2} - \int_0^1 \tan^{-1}(1-x+x^2)\,dx$

$\therefore \quad \int_0^1 \tan^{-1}(1-x+x^2)\,dx = \frac{\pi}{2} - \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx$

Now, $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = 2\int_0^1 \tan^{-1} x\,dx$

$= 2\left[[x\tan^{-1} x]_0^1 - \int_0^1 \frac{x}{1+x^2}dx\right]$

$= 2\left[\frac{\pi}{4} - \frac{1}{2}[\log(1+x^2)]_0^1\right]$

$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right] = \frac{\pi}{2} - \log 2$

$\therefore \quad \int_0^1 \tan^{-1}(1-x+x^2)\,dx = \frac{\pi}{2} - \left(\frac{\pi}{2} - \log 2\right) = \log 2$

Property V

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

and $$\int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \end{cases}$$

Example 12 Evaluate

$$\int_0^{\pi} e^{|\cos x|}\left[2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\,dx.$$ [IIT JEE 2005]

Sol. Let $I = \int_0^{\pi} e^{|\cos x|}\left[2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\,dx$

$= \int_0^{\pi} e^{|\cos x|}\sin x \cdot 2\sin\left(\frac{1}{2}\cos x\right)dx$

$+ \int_0^{\pi} e^{|\cos x|}\cdot 3\cos x\left(\frac{1}{2}\cos x\right)\sin x\,dx$

$\Rightarrow \quad I = I_1 + I_2$ ...(i)

Now, $I_1 = 0$ as $f(\pi - x) = -f(x)$ ...(ii)

and $I_2 = 6\int_0^{\pi/2} e^{\cos x}\sin x\cos\left(\frac{1}{2}\cos x\right)dx$

Now, $I_2 = 6\int_0^1 e^t \cos\left(\frac{1}{2}\right)dt$

[put $\cos x = t \Rightarrow -\sin x\,dx = dt$]

$= 6\left[\frac{e^t}{1+1/4}\left\{\cos\left(\frac{t}{2}\right) + \frac{1}{2}\sin\left(\frac{t}{2}\right)\right\}\right]_0^1$

$= \frac{24}{5}\left[e\cos\left(\frac{1}{2}\right) + \frac{e}{2}\sin\left(\frac{1}{2}\right) - 1\right]$ ...(iii)

From Eq. (i), we get $I = \frac{24}{5}\left[e\cos\left(\frac{1}{2}\right) + \frac{e}{2}\sin\left(\frac{1}{2}\right) - 1\right]$

Property VI

Definite integration of odd and even functions is given as

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even} \end{cases}$$

POINTS TO REMEMBER

(i) If $f(-x) = f(x)$, then $f(x)$ is an even function.

(ii) If $f(-x) = -f(x)$, then $f(x)$ is an odd function.

Example 13 The integral $\int_{-1/2}^{1/2} [x] + \log\left[\frac{1+x}{1-x}\right]dx$ equals [IIT JEE 2002]

(a) $-\frac{1}{2}$ (b) 0 (c) 1 (d) $\log\left(\frac{1}{2}\right)$

Sol. **(a)** Let $I = \int_{-1/2}^{1/2}\left\{[x] + \log\left(\frac{1+x}{1-x}\right)\right\}dx$

$= \int_{-1/2}^{1/2} [x]\,dx + \int_{-1/2}^{1/2} \log\left(\frac{1+x}{1-x}\right)dx$

$= \int_{-1/2}^{1/2} [x]\,dx + 0\left[\because \log\left(\frac{1+x}{1-x}\right) \text{is an odd function}\right]$

$= \int_{-1/2}^{0} [x]\,dx + \int_0^{1/2} [x]\,dx$

$= \int_{-1/2}^{0} (-1)\,dx + \int_0^{1/2} (0)\,dx = -[x]_{-1/2}^0 = -\frac{1}{2}$

Example 14 The value of the integral

$$\int_{-\pi/2}^{\pi/2}\left[x^2 + \log\left(\frac{\pi - x}{\pi + x}\right)\right]\cos x\,dx \text{ is}$$ [IIT JEE 2012]

(a) 0 (b) $\frac{\pi^2}{2} - 4$ (c) $\frac{\pi^2}{2} + 4$ (d) $\frac{\pi^2}{2}$

Sol. **(b)** Let $I = \int_{-\pi/2}^{\pi/2}\left[x^2 + \log\left(\frac{\pi - x}{\pi + x}\right)\right]\cos x\,dx$

$= \int_{-\pi/2}^{\pi/2} x^2\cos x\,dx + \int_{-\pi/2}^{\pi/2} \log\left(\frac{\pi - x}{\pi + x}\right)\cos x\,dx$

$= 2\int_0^{\pi/2} x^2\cos x\,dx + 0 = 2\left[[x^2\sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x\cdot 2x\,dx\right]$

$= 2\left[\frac{\pi^2}{4} - 2\{[x(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} -\cos x\,dx\}\right]$

$= 2\left[\frac{\pi^2}{4} - 2[\sin x]_0^{\pi/2}\right] = 2\left[\frac{\pi^2}{4} - 2\right] = \frac{\pi^2}{2} - 4$

Example 15 Evaluate the definite integration

$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx.$$

[IIT JEE 2002]

Sol. Let $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4}\right)\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{2x}{1+x^2}\right)\right]dx$

$$= \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx - \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}(2\tan^{-1}x)\,dx$$

$$= \frac{\pi}{2}\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx - 0$$

$$\left[\because \frac{x^4}{1-x^4}(2\tan^{-1}x) \text{ is an odd function}\right]$$

$$= 2\cdot\frac{\pi}{2}\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx$$

$$= \pi\int_0^{1/\sqrt{3}}\left(-1 - \frac{1}{1-x^4}\right)dx$$

$$= \frac{\pi}{2}\int_0^{1/\sqrt{3}}\left(-2 + \frac{1}{1-x^2} + \frac{1}{1+x^2}\right)dx$$

$$= \frac{\pi}{2}\left[-2x + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + \tan^{-1}x\right]_0^{1/\sqrt{3}}$$

$$= \frac{\pi}{2}\left[\frac{-2}{\sqrt{3}} + \frac{1}{2}\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + \frac{\pi}{6}\right]$$

$$= \frac{\pi}{12}[\pi + 3\log(2+\sqrt{3}) - 4\sqrt{3}]$$

Example 16 Determine the value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$.

[IIT JEE 2007]

Sol. Let $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$

$$= \int_{-\pi}^{\pi}\frac{2x}{1+\cos^2 x}dx + \int_{-\pi}^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx$$

$$= 0 + 2\int_0^{\pi}\frac{2x\sin x}{1+\cos^2 x}dx$$

$$= 4\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx \quad \ldots(i)$$

$$= 4\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

$$= 4\int_0^{\pi}\frac{(\pi-x)\sin x}{1+\cos^2 x}dx \quad \ldots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = 4\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow \quad I = 2\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx = 4\pi\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}dx$$

$$= -4\pi\,[\tan^{-1}(\cos x)]_0^{\pi/2} = -4\pi\left(0 - \frac{\pi}{4}\right) = \pi^2$$

Example 17 Evaluate $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2-\cos\left(|x| + \frac{\pi}{3}\right)}dx$.

[IIT JEE 2004]

Sol. Let $I = \int_{-\pi/3}^{\pi/3}\frac{\pi\,dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)} + 4\int_{-\pi/3}^{\pi/3}\frac{x^3\,dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)}$

$$= 2\int_0^{\pi/3}\frac{\pi\,dx}{2-\cos\left(|x|+\frac{\pi}{3}\right)} + 0 = 2\pi\int_0^{\pi/3}\frac{dx}{2-\cos\left(x+\frac{\pi}{3}\right)}$$

Put $x + \frac{\pi}{3} = t \Rightarrow dx = dt$

$$\therefore \quad I = 2\pi\int_{\pi/3}^{2\pi/3}\frac{dt}{2-\cos t} = 2\pi\int_{\pi/3}^{2\pi/3}\frac{\sec^2\frac{t}{2}}{1+3\tan^2\frac{t}{2}}dt$$

Put $\tan\frac{t}{2} = u \Rightarrow \sec^2\frac{t}{2}dt = 2du$

$$\therefore \quad I = 2\pi\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{2du}{1+3u^2} = \frac{4\pi}{3}\int_{1/\sqrt{3}}^{\sqrt{3}}\frac{du}{1+3u^2}$$

$$= \frac{4\pi}{3}[\sqrt{3}\tan^{-1}\sqrt{3}u]_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{4\pi}{\sqrt{3}}[\tan^{-1}3 - \tan^{-1}1]$$

$$= \frac{4\pi}{\sqrt{3}}\tan^{-1}\left(\frac{1}{2}\right)$$

Definite Integration of Periodic Functions

Let $f(x)$ be a function with period T, then

(i) $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$

(ii) $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx$

(iii) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx$

(iv) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx$

Example 18 Let $T > 0$ be a fixed real number. Suppose f is continuous function such that for all $x \in R$, $f(x+T) = f(x)$. If $I = \int_0^T f(x)dx$, then the value of $\int_3^{3+3T} f(2x)dx$ is [IIT JEE 2002]

(a) $(3/2)I$ (b) $2I$ (c) $3I$ (d) $6I$

Sol. **(c)** Let $I_1 = \int_3^{3+3T} f(2x)dx$

Put $2x = y$, so that $I_1 = \frac{1}{2}\int_6^{6+6T} f(y)dy$

$$= \frac{1}{2}\cdot 6\int_0^T f(y)dy \qquad [\because f(x) \text{ has period } T]$$

$$= 3I$$

Example 19 Show that $\int_0^{n\pi+v} |\sin x|\,dx = 2n + 1 - \cos v$, where n is a positive integer and $0 \le v < \pi$. [IIT JEE 2004]

Sol. Let $I = \int_0^{n\pi+v} |\sin x|\,dx$

$= \int_0^{v} |\sin x|\,dx + \int_v^{n\pi+v} |\sin x|\,dx$

$= \int_0^{v} \sin x\,dx + n\int_0^{\pi} |\sin x|\,dx$

[$\because$ $|\sin x|$ has a period π]

$= [-\cos x]_0^v + n[-\cos x]_0^{\pi}$

$= 2n + 1 - \cos v$

Inequalities

Here, we will discuss same important inequalities based on definite integral.

(i) If at every point x of an interval $[a, b]$. The inequalities $g(x) \le f(x) \le h(x)$ are fullfilled, then

$$\int_a^b g(x)dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx$$

(ii) If m is the least value (global minimum) and M is the greatest value (global maximum) of the function $f(x)$ on the interval $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

(iii) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b f(x)\,dx$

(iv) If $f^2(x)$ and $g^2(x)$ are integrable on the interval $[a, b]$, then

$$\int_a^b f(x)\,g(x)\,dx \le \sqrt{\left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right)}$$

Example 20 Let $g(x) = \int_0^x f(t)\,dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$, for $t \in [0,1]$ and $0 \le f(t) \le \frac{1}{2}$, for $t \in [1, 2]$. Then, $g(2)$ satisfies the inequality [IIT JEE 2000]

(a) $-\frac{3}{2} \le g(2) < \frac{1}{2}$

(b) $\frac{1}{2} \le g(2) \le \frac{3}{2}$

(c) $\frac{3}{2} < g(2) \le \frac{5}{2}$

(d) $2 < g(2) < 4$

Sol. ***(b)*** We have,

$g(x) = \int_0^x f(t)\,dt$

$\therefore$ $g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$

Now, $\frac{1}{2} \le f(t) \le 1$ for $t \in [0,1]$

$\Rightarrow$ $\int_0^1 \frac{1}{2}dt \le \int_0^1 f(t)\,dt \le \int_0^1 1\,dt$

$\Rightarrow$ $\frac{1}{2} \le \int_0^1 f(t)\,dt \le 1$ ...(i)

Again, $0 \le f(t) \le \frac{1}{2}$ for $t \in [1, 2]$

$\Rightarrow$ $\int_1^2 0\,dt \le \int_1^2 f(t)\,dt \le \int_1^2 \frac{1}{2}\,dt$

$\Rightarrow$ $0 \le \int_1^2 f(t)\,dt \le \frac{1}{2}$ ...(ii)

From Eqs. (i) and (ii), we get

$\frac{1}{2} \le \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \le \frac{3}{2}$

$\Rightarrow$ $\frac{1}{2} \le g(2) \le \frac{3}{2}$

Example 21 Let $f'(x) = \frac{192x^3}{2 + \sin^4(\pi x)}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$ $m \le \int_{1/2}^1 f(x)\,dx \le M$, then the possible values of m and M are [JEE Advanced 2012]

(a) $m = 13, M = 24$ (b) $m = \frac{1}{4}, M = \frac{1}{2}$

(c) $m = -11, M = 0$ (d) $m = 1, M = 12$

Sol. ***(d)*** We have, $f'(x) = \frac{192x^3}{2 + \sin^4(\pi x)}, \forall\, x \in R; f\left(\frac{1}{2}\right) = 0$

Now, $64x^3 \le f'(x) \le 96x^3, \forall\, x \in \left[\frac{1}{2}, 1\right]$

$\int_{1/2}^x 64x^3\,dx \le \int_{1/2}^x f'(x)\,dx \le \int_{1/2}^x 96x^3\,dx$

So, $16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}, \forall\, x \in \left[\frac{1}{2}, 1\right]$

$\int_{1/2}^1 (16x^4 - 1)\,dx \le \int_{1/2}^1 f(x)\,dx \le \int_{1/2}^1 \left(24x^4 - \frac{3}{2}\right)dx$

$\Rightarrow$ $\frac{16}{5}\cdot\frac{31}{32} - \frac{1}{2} \le \int_{1/2}^1 f(x)\,dx \le \frac{24}{5}\cdot\frac{31}{32} - \frac{3}{4}$

$\Rightarrow$ $\frac{26}{10} \le \int_{1/2}^1 f(x)\,dx \le \frac{78}{20}$

$\Rightarrow$ $2.6 \le \int_{1/2}^1 f(x)\,dx \le 3.9$

Do You Know

The term "integral" was invented by the Swiss mathematician Jakob Bernoulli, the older brother of Johann. Both were students of Leibniz and made outstandiing contributioins in the new frontier of calculus. They were also quarrelsome, Johann especially so. Leibniz tried to mediate but being professional diplomat he was not very successful.

Smart Practice

Single Option Correct Type

1. If $[x], \{x\}$ denote the integral and fractional parts of x and $n \in N$, then the value of $\dfrac{\int_0^n [x]\,dx}{\int_0^n \{x\}\,dx}$ is

(a) $\dfrac{1}{n-1}$ (b) $\dfrac{1}{n}$ (c) n (d) $n-1$

2. If $\int_{-1}^{-4} f(x)\,dx = 4$ and $\int_{2}^{-4}(3 - f(x))\,dx = 7$, then the value of $\int_{-2}^{1} f(-x)\,dx$, is

(a) -29 (b) -3
(c) 5 (d) None of these

3. If $f(x) = \dfrac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} x\, g\{x(1-x)\}\,dx$ and $I_2 = \int_{f(-a)}^{f(a)} g\{x(1-x)\}\,dx$, then the value of $\dfrac{I_2}{I_1}$ is

(a) -3 (b) -1 (c) 2 (d) 3

4. If f is an odd function, then the value of $\int_{-a}^{a} \dfrac{f(\sin x)}{f(\cos x) + f(\sin^2 x)}\,dx$ is

(a) $2\int_0^a \dfrac{f(\sin x)}{f(\cos x) + f(\sin^2 x)}\,dx$ (b) 0
(c) a (d) None of these

5. $\int_{-\pi/2}^{\pi/2} \dfrac{e^{|\sin x|}\cos x}{1+e^{\tan x}}\,dx$ is equal to

(a) $e+1$ (b) $1-e$
(c) $e-1$ (d) None of these

6. If $f(x) = \begin{cases} e^{\cos x}\sin x, & \text{for } |x| \le 2 \\ 2, & \text{otherwise} \end{cases}$, then $\int_{-2}^{3} f(x)\,dx$ is equal to

(a) 0 (b) 1 (c) 2 (d) 3

7. The value of $\int_{-1}^{3/2} |x \sin \pi x|\,dx$ is

(a) $(3\pi+1)/\pi^2$ (b) 3π
(c) π^2 (d) $\dfrac{3\pi - 1}{\pi^2}$

8. The value of $\int_{-\pi}^{\pi} \dfrac{\cos^2 x}{1+a^x}\,dx$, $a > 0$ is

(a) π (b) $a\pi$ (c) $\pi/2$ (d) 2π

9. For $n > 0$, $\int_0^{2\pi} \dfrac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$ is equal to

(a) π (b) π^2 (c) 2π (d) 0

10. The value of $\int_0^{\pi} \dfrac{x\,dx}{1+\cos\alpha \sin x}$, $0 < \alpha < \pi$ is

(a) $\dfrac{\pi}{\sin\alpha}$ (b) $\dfrac{\sin\alpha}{\pi}$
(c) $\dfrac{\alpha\pi}{\sin\alpha}$ (d) $\dfrac{\sin\alpha}{\alpha\pi}$

11. For any integer n, the integral $\int_0^{\pi} e^{\cos^2 x}\cos^3\{(2n+1)x\}\,dx$ has the value

(a) π (b) 1
(c) 0 (d) None of these

12. Let $f(x) = x - [x]$, for every real number x, where $[x]$ is the integral part of x. Then, $\int_{-1}^{1} f(x)\,dx$ is

(a) 1 (b) 2 (c) 0 (d) $-\dfrac{1}{2}$

13. The value of $\int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]\,dx$ is

(a) 0 (b) 3 (c) 4 (d) 1

14. The value of $\int_{-1}^{1} \left(\cos^{-1} x + \dfrac{x^7 - 3x^5 + 7x^3 - x}{\cos^2 x}\right)dx$ is

(a) $\pi/2$ (b) 0 (c) 2π (d) π

15. If I is the greatest of the definite integrals

$I_1 = \int_0^1 e^{-x}\cos^2 x\,dx$, $I_2 = \int_0^1 e^{-x^2}\cos^2 x\,dx$

$I_3 = \int_0^1 e^{-x^2}\,dx$, $I_4 = \int_0^1 e^{-x^2/2}\,dx$, then

(a) $I = I_1$ (b) $I = I_2$ (c) $I = I_3$ (d) $I = I_4$

One or More than One Option Correct Type

16. If $\int_a^b \dfrac{f(x)}{f(x) + f(a+b-x)}\,dx = 10$, then

(a) $a = 2, b = 22$ (b) $a = -5, b = 15$
(c) $a = -10, b = 10$ (d) $a = -2, b = 10$

17. If $f(2-x) = f(2+x)$ and $f(4-x) = f(4+x)$ for all x and $f(x)$ is a function for which $\int_0^2 f(x)\,dx = 5$, then $\int_1^{50} f(x)\,dx$ is equal to

(a) 125 (b) $\int_{-4}^{46} f(x)\,dx$ (c) $\int_1^{51} f(x)\,dx$ (d) $\int_2^{52} f(x)\,dx$

Linked Comprehension Type

Directions (Q. Nos. 18-20)

$$f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t\cos x)\, f(t)\,dt$$

18. The range of $f(x)$ is

(a) $\left[\frac{-\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ (b) $\left[\frac{-\sqrt{5}}{3}, \frac{\sqrt{5}}{3}\right]$

(c) $\left[\frac{-\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$ (d) None of these

19. $f(x)$ is not invertible for

(a) $x \in \left[-\frac{\pi}{2} - \tan^{-1}2, \frac{\pi}{2} - \tan^{-1}2\right]$

(b) $x \in \left[\tan^{-1}\frac{1}{2}, \pi + \tan^{-1}\frac{1}{2}\right]$

(c) $x \in [\pi + \cot^{-1}2, 2\pi + \cot^{-1}2]$

(d) None of the above

20. The value of $\int_0^{\pi/2} f(x)\,dx$ is

(a) 1 (b) -2 (c) -1 (d) 2

Directions (Q. Nos. 21-24) Let $f(x)$ and $\phi(x)$ are two continuous functions on R satisfying $\phi(x) = \int_a^x f(t)\,dt$, $a \neq 0$ and another continuous function $g(x)$ satisfying $g(x + \alpha) + g(x) = 0, \forall\, x \in R$, $\alpha > 0$ and $\int_b^{2k} g(t)\,dt$ is independent of b.

21. If $f(x)$ is an odd function, then

(a) $\phi(x)$ is also an odd function
(b) $\phi(x)$ is an even function
(c) $\phi(x)$ is neither an even nor an odd function
(d) for $\phi(x)$ to be an even function, it must satisfy $\int_0^a f(x)\,dx = 0$

22. If $f(x)$ is an even function, then

(a) $\phi(x)$ is also an even function
(b) $\phi(x)$ is an odd function
(c) if $f(a - x) = -f(x)$, then $\phi(x)$ is an even function
(d) if $f(a - x) = -f(x)$, then $\phi(x)$ is an odd function

23. Least positive value of c, if c, k, b are in AP, is

(a) 0 (b) 1 (c) α (d) 2α

24. If m, n are even integers and p, $q \in R$, then $\int_{p+m\alpha}^{q+n\alpha} g(t)\,dt$ is equal to

(a) $\int_p^q g(x)\,dx$

(b) $(n - m)\int_0^\alpha g(x)dx$

(c) $\int_p^q g(x)dx + (n - m)\int_0^\alpha g(2x)\,dx$

(d) $\int_p^q g(x)dx + (n - m)\int_0^\alpha g(x)\,dx$

Single Integer Type

25. A continuous real function f satisfies $f(2x) = 3f(x), \forall\, x \in R$. If $\int_0^1 f(x)\,dx = 1$, then the value of definite integral $\int_1^2 f(x)\,dx$ is ..... .

26. The value of $\frac{2^{2010}}{1005} \cdot \frac{\int_0^1 x^{1004}(1-x)^{1004}\,dx}{\int_0^1 x^{1004}(1-x^{2010})^{1004}\,dx}$ is ..... .

27. Let $f : R \to R$ be given by

$$f(x) = \begin{cases} |x - [x]|, & \text{when } [x] \text{ is odd} \\ |x - [x] - 1|, & \text{when } [x] \text{ is even} \end{cases}$$

where, $[.]$ denotes the greatest integer function, then $\int_{-2}^4 f(x)\,dx$ is equal to ..... .

28. If $f(x) = \frac{e^x + 1}{e^x - 1}$, $\int_0^1 x^3\left(\frac{e^x + 1}{e^x - 1}\right) = 4$. Then, $\int_{-1}^1 t^3 f(t)\,dt$ is equal to ..... .

29. If $\int_1^4 \log_e[x]\,dx = \log_e \lambda$, then λ = ...

Match the Columns

30.

Column I	Column II
A. If $f(x)$ is an integrable function for $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $I_1 = \int_{\pi/6}^{\pi/3} \sec^2\theta\, f(2\sin 2\theta)\,d\theta$ and $I_2 = \int_{\pi/6}^{\pi/3} \operatorname{cosec}^2\theta f(2\sin 2\theta)\,d\theta$, then $I_1/I_2 =$	p. 3
B. If $f(x + 1) = f(3 + x), \forall\, x$ and the value of $\int_a^{a+b} f(x)\,dx$ is independent of a, then the value of b can be	q. 1
C. The value of $\int_1^4 \frac{\tan^{-1}[x^2]\,dx}{\tan^{-1}[x^2] + \tan^{-1}[25 + x^2 - 10x]}$ (where $[\cdot]$ denotes the greatest integer function) is	r. 2
D. If $I = \int_0^2 \sqrt{x + \sqrt{x + \sqrt{x + \ldots \infty}}}\,dx$ (where $x > 0$), then $[I]$ is equal to (where $[\cdot]$ denotes the greatest integer function)	s. 4

Answers with Explanation

1. ***(d)*** We have, $I_1 = \int_0^n [x]\,dx$

$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \ldots + \int_{n-1}^n (n-1)\,dx$

$= 1(2-1) + 2(3-2) + 3(4-3) + \ldots + (n-1)\{n-(n-1)\}$

$= 1 + 2 + 3 + \ldots + (n-1) = \frac{n(n-1)}{2}$

and $I_2 = \int_0^n \{x\}\,dx = \int_0^n (x - [x])\,dx = \frac{n}{2}$

$\therefore \quad \frac{I_1}{I_2} = n - 1$

2. ***(a)*** We have, $\int_2^{-4} (3 - f(x))\,dx = 7$

$\Rightarrow \quad \int_2^{-4} 3\,dx - \int_2^{-4} f(x)\,dx = 7$

$\Rightarrow \quad 3(-4-2) - \int_2^{-4} f(x)\,dx = 7$

$\Rightarrow \quad \int_2^{-4} f(x)\,dx = -25 \Rightarrow \int_{-4}^2 f(x)\,dx = 25$

$\Rightarrow \quad \int_{-4}^{-1} f(x)\,dx + \int_{-1}^2 f(x)\,dx = 25$

$\Rightarrow \quad -4 + \int_{-1}^2 f(x)\,dx = 25 \qquad \left[\because \int_{-1}^{-4} f(x)\,dx = 4\right]$

$\Rightarrow \quad \int_{-1}^2 f(x)\,dx = 29$

$\Rightarrow \quad -\int_1^{-2} f(-t)\,dt = 29, \text{ where } t = -x$

$\Rightarrow \quad \int_{-2}^1 f(-t)\,dt = -29 \Rightarrow \int_{-2}^1 f(-x)\,dx = -29$

3. ***(c)*** We have, $f(x) = \frac{e^x}{1+e^x}$

$\Rightarrow \quad f(a) + f(-a) = \frac{e^a}{1+e^a} + \frac{e^{-a}}{1+e^{-a}}$

$= \frac{e^a}{1+e^a} + \frac{1}{1+e^a} = 1$

Now, $I_1 = \int_{f(-a)}^{f(a)} x\, g\,\{x(1-x)\}\,dx$

$I_1 = \int_{f(-a)}^{f(a)} (1-x)\, g\,\{(1-x)(1-(1-x))\}dx$

$[\because f(a) + f(-a) = 1, \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx]$

$I_1 = \int_{f(-a)}^{f(a)} g\,\{(1-x)x\}\,dx - I_1$

$2I_1 = I_2 \Rightarrow I_2/I_1 = 2$

4. ***(b)*** Let $\phi(x) = \frac{f(\sin x)}{f(\cos x) + f(\sin^2 x)}$

$\therefore \quad \phi(-x) = \frac{f(\sin(-x))}{f(\cos(-x)) + f(\sin^2(-x))}$

$= \frac{f(-\sin x)}{f(\cos x) + f(\sin^2 x)}$

$= \frac{-f(\sin x)}{f(\cos x) + f(\sin^2 x)} \quad [\because f(x) \text{ is an odd function}]$

$= -\phi(x)$

$\therefore \quad I = \int_{-a}^{a} \frac{f(\sin x)}{f(\cos x) + f(\sin^2 x)}\,dx = 0$

5. ***(c)*** Let $I = \int_{-\pi/2}^{\pi/2} \frac{e^{|\sin x|}\cos x}{1+e^{\tan x}}\,dx \quad \ldots(i)$

$\therefore \quad I = \int_{-\pi/2}^{\pi/2} \frac{e^{|\sin(-x)|}\cos(-x)}{1+e^{\tan(-x)}}\,dx$

$= \int_{-\pi/2}^{\pi/2} \frac{e^{|\sin x|}\cos x}{1+e^{-\tan x}}\,dx \quad \ldots(ii)$

On adding Eqs. (i) and (ii), we get

$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \left(\frac{e^{|\sin x|}\cos x}{1+e^{\tan x}} + \frac{e^{|\sin x|}\cos x}{1+e^{-\tan x}}\right)dx$

$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} e^{|\sin x|}\cos x\,dx \Rightarrow 2I = 2\int_0^{\pi/2} e^{|\sin x|}\cos x\,dx$

$\Rightarrow \quad I = \int_0^{\pi/2} e^{\sin x}\cos x\,dx = [e^{\sin x}]_0^{\pi/2} = e - 1$

6. ***(c)*** Given, $f(x) = \begin{cases} e^{\cos x}\sin x, & \text{for } |x| \le 2 \\ 2, & \text{otherwise} \end{cases}$

$\therefore \quad \int_{-2}^3 f(x)\,dx = \int_{-2}^2 f(x)\,dx + \int_2^3 f(x)\,dx$

$= \int_{-2}^2 e^{\cos x}\sin x\,dx + \int_2^3 2\,dx$

$= 0 + 2[x]_2^3$ [$\because e^{\cos x}\sin x$ is an odd function]

$= 2[3-2] = 2$

7. ***(a)*** Let $I = \int_{-1}^{3/2} |x\sin \pi x|\,dx$

Since, $|x\sin\pi x| = \begin{cases} x\sin\pi x, & -1 < x \le 1 \\ -x\sin\pi x, & 1 < x < \frac{3}{2} \end{cases}$

$\therefore \quad I = \int_{-1}^1 x\sin\pi x\,dx + \int_1^{3/2} -x\sin\pi x\,dx$

$I = 2\int_0^1 x\sin\pi x\,dx + \int_1^{3/2} -x\sin\pi x\,dx$

$= 2\left[\frac{-x\cos\pi x}{\pi}\right]_0^1 - 2\int_0^1 1\cdot\left(-\frac{\cos\pi x}{\pi}\right)dx$

$- \left\{\left[\frac{-x\cos\pi x}{\pi}\right]_1^{3/2} - \int_1^{3/2}\left(\frac{-\cos\pi x}{\pi}\right)dx\right\}$

$= 2\left(\frac{1}{\pi}\right) + \frac{2}{\pi}\left[\frac{\sin\pi x}{\pi}\right]_0^1 - \left(-\frac{1}{\pi}\right) - \frac{1}{\pi}\left[\frac{\sin\pi x}{\pi}\right]_1^{3/2}$

$= \frac{2}{\pi} + \frac{2}{\pi^2}(0-0) + \frac{1}{\pi} + \frac{1}{\pi^2}(1-0)$

$= \frac{3}{\pi} + \frac{1}{\pi^2} = \frac{3\pi+1}{\pi^2}$

8. ***(c)*** Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx \quad \ldots(i)$

$= \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}}\,dx = \int_{-\pi}^{\pi} \frac{a^x}{1+a^x}\cos^2 x\,dx \quad \ldots(ii)$

On adding Eqs. (i) and (ii), we get

$2I = \int_{-\pi}^{\pi} \left(\frac{1+a^x}{1+a^x}\right)\cos^2 x\,dx$

$= \int_{-\pi}^{\pi} \cos^2 x\,dx = 2\int_0^{\pi} \cos^2 x\,dx$

$$= 2\int_0^{\pi} \frac{1+\cos 2x}{2}\,dx = \int_0^{\pi}(1+\cos 2x)\,dx$$

$$= \left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = \pi \Rightarrow I = \frac{\pi}{2}$$

9. ***(b)*** Let $I = \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$ ...(i)

$$\Rightarrow I = \int_0^{2\pi} \frac{(2\pi - x)\,[\sin(2\pi - x)]^{2n}}{[\sin(2\pi - x)]^{2n} + [\cos(2\pi - x)]^{2n}}\,dx$$

$$= \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$$

$$= \int_0^{2\pi} \frac{2\pi \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx - \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$$

$$\Rightarrow \quad 2I = 2\pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$$

$$\Rightarrow \quad I = \pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$$

$$\Rightarrow \quad I = 2\pi \int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx \quad \text{...(ii)}$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n}\left(\frac{\pi}{2} - x\right)}{\sin^{2n}\left(\frac{\pi}{2} - x\right) + \cos^{2n}\left(\frac{\pi}{2} - x\right)}\,dx$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x}\,dx \quad \text{...(iii)}$$

On adding Eqs. (ii) and (iii), we get

$$2I = 4\pi \int_0^{\pi/2} 1\,dx \Rightarrow 2I = 4\pi \cdot \frac{\pi}{2} \Rightarrow I = \pi^2$$

10. ***(c)*** Let $I = \int_0^{\pi} \frac{x}{1 + \cos\alpha \sin x}\,dx$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi - x}{1 + \cos\alpha \sin(\pi - x)}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{\pi - x}{1 + \cos\alpha \sin x}\,dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \pi \int_0^{\pi} \frac{dx}{1 + \cos\alpha \sin x}$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sec^2 x/2}{1 + \tan^2 x/2 + 2\cos\alpha \tan x/2}\,dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}\,dx = 2\,dt$

$$\therefore \quad 2I = \pi \int_0^{\infty} \frac{2dt}{1 + t^2 + 2t\cos\alpha}$$

$$\Rightarrow \quad I = \pi \int_0^{\infty} \frac{dt}{(t + \cos\alpha)^2 + \sin^2\alpha}$$

$$= \frac{\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{t + \cos\alpha}{\sin\alpha}\right)\right]_0^{\infty}$$

$$= \frac{\pi}{\sin\alpha}\left[\tan^{-1}\infty - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)\right] = \frac{\alpha\pi}{\sin\alpha}$$

11. ***(c)*** Let $I = \int_0^{\pi} e^{\cos^2 x} \cos^3\{(2n+1)\,x\}\,dx$

Again, let $f(x) = e^{\cos^2 x}\cos^3\{(2n+1)x\}$

$$\Rightarrow \quad f(\pi - x) = e^{\cos^2(\pi - x)}\cos^3\{(2n+1)(\pi - x)\}$$

$$= -e^{\cos^2 x}\cos^3\{(2n+1)x\} = -f(x)$$

$$\therefore \quad I = 0$$

12. ***(a)*** Let $I = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} \{x - [x]\}\,dx = \int_{-1}^{1} x\,dx - \int_{-1}^{1} [x]\,dx$

$$= 0 - \int_{-1}^{1} [x]\,dx \qquad [\because x \text{ is an odd function}]$$

$$= -\left[\int_{-1}^{0} [x]\,dx + \int_0^1 [x]\,dx\right]$$

$$= -\left[\int_{-1}^{0} (-1)\,dx + \int_0^1 (0)\,dx\right] = -[-x]_{-1}^{0} = 1$$

13. ***(c)*** Let $I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]\,dx$

$$= \int_{-2}^{0} [(x+1)^3 + 2 + (x+1)\cos(x+1)]\,dx$$

Put $x + 1 = t \Rightarrow dx = dt$

$$\therefore \quad I = \int_{-1}^{1} [t^3 + 2 + t\cos t]\,dt$$

$$= \int_{-1}^{1} t^3\,dt + 2\int_{-1}^{1} dt + \int_{-1}^{1} t\cos t\,dt$$

$$= 0 + 2 \cdot 2[x]_0^1 + 0 \qquad [\because t^3, t\cos t \text{ are odd functions}]$$

$$= 4$$

14. ***(d)*** Let $f(x) = \frac{x^7 - 3x^5 + 7x^3 - x}{\cos^2 x}$

Now, $f(-x) = -f(x)$. So, $\int_{-1}^{1} f(x)\,dx = 0$

Let $g(x) = \cos^{-1} x$, then

$$I_1 = \int_{-1}^{1} \cos^{-1} x\,dx = \int_{-1}^{1} \cos^{-1}(-x)\,dx$$

$$= \int_{-1}^{1} (\pi - \cos^{-1} x)\,dx = 2\pi - I_1$$

Thus, $I = I_1 = \pi$

15. ***(d)*** For $0 < x < 1$, we have $\frac{1}{2}x^2 < x^2 < x$

$$\Rightarrow \quad -x^2 > -x \Rightarrow e^{-x^2} > e^{-x}$$

Hence, $\int_0^1 e^{-x^2}\cos^2 x\,dx > \int_0^1 e^{-x^2}\cos^2 x\,dx$

Also, $\cos^2 x \le 1$.

Therefore, $\int_0^1 e^{-x^2}\cos^2 x\,dx \le \int_0^1 e^{-x^2}\,dx < \int_0^1 e^{-x^2/2}\,dx$

$$\therefore \quad I = I_4$$

16. ***(a, b, c)*** Let $I = \int_a^b \frac{f(x)}{f(x) + f(a+b-x)}\,dx$ ...(i)

$$\therefore \quad I = \int_a^b \frac{f(a+b-x)}{f(a+b-x) + f(a+b-(a+b-x))}\,dx$$

$$= \int_a^b \frac{f(a+b-x)}{f(x) + f(a+b-x)}\,dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_a^b 1\,dx \Rightarrow 2I = [x]_a^b \Rightarrow 2I = b - a \Rightarrow I = \frac{b-a}{2}$$

Hence, options (a), (b), (c) satisfy the above condition.

17. ***(a, b, c)*** We have, $f(2 - x) = f(2 + x)$, $f(4 - x) = f(4 + x)$

or $f(4 + x) = f(4 - x) = f(2 + (2 - x)) = f(2 - (2 - x)) = f(x)$

Thus, the period of $f(x)$ is 4.

$$\int_0^{50} f(x)\,dx = \int_0^{48} f(x)\,dx + \int_{48}^{50} f(x)\,dx = 12\int_0^4 f(x)\,dx + \int_0^2 f(x)\,dx$$

[In second integral, replacing x by $x + 48$ and then using $f(x) = f(x + 48)$]

$$= 12\left[\int_0^2 f(x)\,dx + \int_0^2 f(4 - x)\,dx\right] + 5$$

$$= 12\left[\int_0^2 f(x)\,dx + \int_0^2 f(4 - x)\,dx\right] + 5$$

$$= 24\int_0^2 f(x)\,dx + 5 = 24 \times 5 + 5 = 125$$

Also, $\int_{-4}^{46} f(x)\,dx = \int_{-4}^{-2} f(x)\,dx + \int_{-2}^{-2+48} f(x)\,dx$

$$= \int_0^2 f(x + 4)\,dx + 12\int_0^4 f(x)\,dx$$

$$= \int_0^2 f(x)\,dx + 24\int_0^2 f(x)\,dx$$

$$= 5 + 24 \times 5 = 125$$

Again, $\int_2^{52} f(x)\,dx = \int_2^4 f(x)\,dx + \int_4^{4+48} f(x)\,dx$

$$= \int_0^2 f(4 - x)\,dx + 12\int_0^4 f(x)\,dx$$

$$= \int_0^2 f(4 + x)\,dx + 24\int_0^2 f(x)\,dx$$

$$= \int_0^2 f(x)\,dx + 24\int_0^2 f(x)\,dx = 5 + 24 \times 5 = 125$$

Again, $\int_1^{51} f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^{3+48} f(x)\,dx$

$$= \int_1^3 f(x)\,dx + 12\int_0^4 f(x)\,dx$$

$$= \int_0^2 f(x + 1)\,dx + 24\int_0^2 f(x)\,dx \neq 125$$

Solution (Q. Nos. 18-20)

$$f(x) = \sin x + \sin x \int_{-\pi/2}^{\pi/2} f(t)\,dt + \cos x \int_{-\pi/2}^{\pi/2} t\, f(t)\,dt$$

$$= \sin x\left(1 + \int_{-\pi/2}^{\pi/2} f(t)\,dt\right) + \cos x \int_{-\pi/2}^{\pi/2} t\, f(t)\,dt$$

$$= A\sin x + B\cos x$$

Thus, $A = 1 + \int_{-\pi/2}^{\pi/2} f(t)\,dt = 1 + \int_{-\pi/2}^{\pi/2} (A\sin t + B\cos t)\,dt$

$$= 1 + A\int_{-\pi/2}^{\pi/2} \sin t\,dt + B\int_{-\pi/2}^{\pi/2} \cos t\,dt$$

$$= 1 + A \times 0 + 2B\int_0^{\pi/2} \cos t\,dt = 1 + 2B\,[\sin t]_0^{\pi/2}$$

$\therefore$ $A = 1 + 2B$ ...(i)

$$B = \int_{-\pi/2}^{\pi/2} t\, f(t)\,dt = \int_{-\pi/2}^{\pi/2} t(A\sin t + B\cos t)\,dt$$

$$= A\int_{-\pi/2}^{\pi/2} t\sin t\,dt + B\int_{-\pi/2}^{\pi/2} t\cos t\,dt$$

$$= 2A\int_0^{\pi/2} t\sin t\,dt + 0 = 2A\,[-t\cos t + \sin t]_0^{\pi/2}$$

$\therefore$ $B = 2A$ ...(ii)

From Eqs. (i) and (ii), we get

$$A = -\frac{1}{3} \quad \text{and} \quad B = -\frac{2}{3}$$

$\therefore$ $f(x) = -\frac{1}{3}(\sin x + 2\cos x)$

Now, $f(x) = -\frac{1}{3}(\sin x + 2\cos x)$

$$= \frac{-\sqrt{5}}{3}\sin(x + \tan^{-1}2) = \frac{-\sqrt{5}}{3}\cos\left(x - \tan^{-1}\frac{1}{2}\right)$$

Thus, the range of $f(x)$ is $\left[\frac{-\sqrt{5}}{3}, \frac{\sqrt{5}}{3}\right]$.

$\therefore$ $f(x)$ is invertible, if

$$-\pi \le x - \tan^{-1}\frac{1}{2} \le 0$$

$\Rightarrow$ $$-\frac{\pi}{2} - \tan^{-1}2 \le x \le \frac{\pi}{2} - \tan^{-1}2$$

$\Rightarrow$ $$0 \le x - \tan^{-1}\frac{1}{2} \le \pi$$

$\Rightarrow$ $$\tan^{-1}\frac{1}{2} \le x \le \pi + \tan^{-1}\frac{1}{2}$$

$\Rightarrow$ $$\pi \le x - \tan^{-1}\frac{1}{2} \le 2\pi$$

$\Rightarrow$ $$x \in [\pi + \cot^{-1}2, 2\pi + \cot^{-1}2]$$

Again, $\int_0^{\pi/2} f(x)\,dx = -\frac{1}{3}\int_0^{\pi/2} (\sin x + 2\cos x)\,dx$

$$= -\frac{1}{3}[-\cos x + 2\sin x]_0^{\pi/2} = -\frac{1}{3}[2 + 1]$$

$$= -1$$

18. ***(b)*** **19.** ***(d)*** **20.** ***(c)***

21. ***(b)*** $f(x)$ is an odd function. Thus,

$$f(x) = -f(-x)\ ;\ \phi(-x) = \int_a^{-x} f(t)\,dt$$

Put $t = -y$

$\therefore$ $\phi(-x) = \int_{-a}^{x} f(-t)(-dt) = \int_{-a}^{x} f(t)\,dt$

$$= \int_{-a}^{a} f(t)\,dt + \int_a^x f(t)\,dt = 0 + \int_a^x f(t)\,dt = \phi(x)$$

22. ***(d)*** If $f(x)$ is an even function, then

$$\phi(-x) = -\int_{-a}^{x} f(t)\,dt = -\int_{-a}^{a} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= -2\int_0^a f(t)\,dt - \int_a^x f(t)\,dt \quad [\text{as } f(x) \text{ is an even function}]$$

Now, $\int_0^a f(t)\,dt = \int_0^a f(a - t)\,dt$

$$= \int_0^a -f(t)\,dt \qquad [\text{using } f(a - x) = -f(x)]$$

$\Rightarrow$ $\int_0^a f(t)\,dt = 0 \Rightarrow \phi(-x) = -\int_a^x f(t)\,dt = -f(x)$

Thus, $\phi(x)$ is an odd function.

23. ***(d)*** $g(x + \alpha) + g(x) = 0$

$\Rightarrow$ $g(x + 2\alpha) + g(x + \alpha) = 0 \Rightarrow g(x + 2\alpha) = g(x)$

Thus, $g(x)$ is periodic with period 2α.

$\therefore$ $\int_b^{2k} g(t)dt = \int_b^{b+c} g(x)\,dx$ $[\because b, k, c \text{ are in AP}]$

This is independent of b. Then, c has least value 2α.

24. *(d)* $\int_{p+m\alpha}^{q+n\alpha} g(t)\,dt = \int_{p+m\alpha}^{p} g(x)\,dx + \int_{p}^{q} g(x)dx + \int_{q}^{q+n\alpha} g(x)\,dx$

$= -m\int_0^{\alpha} g(x)dx + \int_p^q g(x)\,dx + n\int_0^{\alpha} g(x)\,dx$

$= \int_p^q g(x)\,dx + (n-m)\int_0^{\alpha} g(x)\,dx$

25. *(5)* We have, $f(2x) = 3f(x)$ ...(i)

and $\int_0^1 f(x)\,dx = 1$ ...(ii)

From Eqs. (i) and (ii), we get $\frac{1}{3}\int_0^1 f(2x)\,dx = 1$

Put $2x = t \Rightarrow 2dx = dt$

Then, $\frac{1}{6}\int_0^2 f(t)\,dt = 1 \Rightarrow \int_0^2 f(t)\,dt = 6$

$\Rightarrow \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 6 \Rightarrow \int_1^2 f(t)\,dt = 6 - \int_0^1 f(t)\,dt$

$\Rightarrow \int_1^2 f(t)\,dt = 6 - 1 \Rightarrow \int_1^2 f(t)\,dt = 5$

26. *(4)* $I_1 = \int_0^1 x^{1004}(1-x)^{1004}\,dx$

$= 2\int_0^{1/2} x^{1004}(1-x)^{1004}\,dx$ ...(i)

Now, $I_2 = \int_0^1 x^{1004}(1-x^{2010})^{1004}\,dx$

Put $x^{1005} = t$ or $1005\,x^{1004}\,dx = dt$

$\therefore \quad I_2 = \frac{1}{1005}\int_0^1 (1-t^2)^{1004}\,dt$

$= \frac{1}{1005}\int_0^1 (t\,(2-t))^{1004}\,dt = \frac{1}{1005}\int_0^1 t^{1004}\,(2-t)^{1004}\,dt$

Now, put $t = 2y$ or $dt = 2dy$

$\therefore \quad I_2 = \frac{1}{1005}\int_0^{1/2} (2y)^{1004}\,(2-2y)^{1004}\,2\,dy$

$= \frac{1}{1005}\,2\cdot 2^{1004}\cdot 2^{1004}\int_0^{1/2} y^{1004}\,(1-y)^{1004}\,dy$

$= \frac{1}{1005}\cdot 2^{2009}\int_0^{1/2} y^{1004}\,(1-y)^{1004}\,dy$

$= \frac{2^{2008}}{1005}\,I_1$

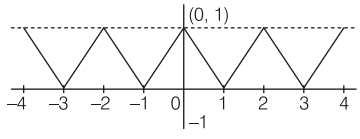
$\therefore \quad \frac{I_1}{I_2} = \frac{1005}{2^{2008}} \Rightarrow \frac{2^{2010}}{1005}\cdot\frac{I_1}{I_2} = 4$

27. *(3)* We have, $f(x) = \begin{cases} |x-[x]|, & \text{when } [x] \text{ is odd} \\ |x-[x]-1|, & \text{when } [x] \text{ is even} \end{cases}$

$= \begin{cases} |x-[x]|, & \text{when } [x] \text{ is odd} \\ |\{x\}-1|, & \text{when } [x] \text{ is even} \end{cases}$

$= \begin{cases} \{x\}, & \text{when } [x] \text{ is odd} \\ 1-\{x\}, & \text{when } [x] \text{ is even} \end{cases}$

The graph of $f(x)$ is as below



$\therefore \quad \int_{-2}^{4} f(x)\,dx = 6\int_{-2}^{-1} f(x)\,dx = 6\times\frac{1}{2} = 3$

28. *(8)* Let $g(x) = x^3 f(x)$.

Then, $g(-x) = (-x)^3 f(-x)$

$\Rightarrow \quad g(-x) = -x^3\left(\frac{e^{-x}+1}{e^{-x}-1}\right) = -x^3\left(\frac{1+e^x}{1-e^x}\right)$

$= -x^3\left[-\left(\frac{e^x+1}{e^x-1}\right)\right] = x^3\cdot\frac{e^x+1}{e^x-1} = g(x)$

$\Rightarrow g(x)$ is an even function.

Hence, $\int_{-1}^{1} t^3 f(t)\,dt = 2\int_0^1 t^3 f(t)\,dt = 2\times 4 = 8$

29. *(6)* We have, $I = \int_1^4 \log_e [x]\,dx$

$= \int_1^2 \log_e [x]\,dx + \int_2^3 \log_e [x]\,dx + \int_3^4 \log_e [x]\,dx$

$= \int_1^2 \log_e 1\,dx + \int_2^3 \log_e 2\,dx + \int_3^4 \log_e 3\,dx$

$= 0 + \log_e 2\,[x]_2^3 + \log_e 3\,[x]_3^4$

$= \log_e 2 + \log_e 3 = \log_e 6$

$\therefore \quad \lambda = 6$

30. A $\rightarrow$ q; B $\rightarrow$ r,s; C $\rightarrow$ p; D $\rightarrow$ p

A. $I_1 = \int_{\pi/6}^{\pi/3} \sec^2\theta\, f(2\sin 2\theta)\,d\theta$

Applying property $\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx$

$I_1 = \int_{\pi/6}^{\pi/3} \sec^2\left(\frac{\pi}{2}-\theta\right) f\left[2\sin 2\left(\frac{\pi}{2}-\theta\right)\right]d\theta$

$= \int_{\pi/6}^{\pi/3} \operatorname{cosec}^2\theta\, f(2\sin 2\theta)\,d\theta = I_2$

B. $f(x+1) = f(x+3)$ or $f(x) = f(x+2)$

Thus, $f(x)$ is periodic with period 2.

Then, $\int_a^{a+b} f(x)\,dx$ is independent of a, for which b is multiple of 2.

Thus, $b = 2, 4, 6$

C. Let $I = \int_1^4 \frac{\tan^{-1}[x^2]}{\tan^{-1}[x^2] + \tan^{-1}[25 + x^2 - 10x]}$ ...(i)

Applying $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, we get

$I = \int_1^4 \frac{\tan[(5-x)^2]}{\tan^{-1}[(5-x)^2] + \tan^{-1}[x^2]}\,dx$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_1^4 dx \Rightarrow 2I = 3 \Rightarrow I = 3/2$

D. Let $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\infty}}} = \sqrt{x+y}$

$\Rightarrow y^2 - y - x = 0$

$\Rightarrow \quad y = \frac{1 \pm \sqrt{1+4x}}{2\cdot 1} = \frac{1+\sqrt{1+4x}}{2}$ [$\because y > 1$]

$\therefore \quad I = \int_0^2 \frac{1+\sqrt{1+4x}}{2}\,dx = \left[\frac{x}{2} + \frac{(1+4x)^{3/2}}{\frac{3}{2}\times 2\times 4}\right]_0^2$

$= \left[\left(1+\frac{27}{12}\right) - \left(0+\frac{1}{12}\right)\right] = 1 + \frac{26}{12} = \frac{19}{6}$

$\therefore \quad [I] = 3$

PERSONALITIES SPECTRUM

SIR ISAAC NEWTON (1642-1727)

LIFE HISTORY

Sir Isaac Newton was an English Physicist and Mathematician. He was born in Woolsthorpe, Lincolnshire, England. He is widely recognised as one of the most influential scientists of all time. His book "Mathematical Principles of Natural Philosophy" laid the foundation of classical mechanics. Newton is remembered for the contribution in the discovery of gravity and the story is related to the apple tree. Newton built the first practical reflecting telescope and developed a theory of colour based on the observation that a prism decomposes white light in many colours of the visible spectrum.

Young Isaac was not an exceptional student. He enjoyed spending much time making contraptions such as a clock which was powered by water and various other inventions. From the age of about twelve to seventeen, Newton was educated at the King's School Granthan. In June 1661, he was admitted to Trinity College, Cambridge. He was awarded a scholarship in 1664 for his further studies. Although he had been undistinguished as a Cambridge student but his private studies at home over the two years saw the development of his theories on calculus, optics and law of gravitation.

ACHIEVEMENTS

He was elected as a fellow of Trinity in 1667. After getting the degree of M.A., he was elected as a fellow of the Royal Society in 1672. He was appointed Lucasian Professor of Mathematics in 1669.

CONTRIBUTION

Newton's work in Mathematics has been said "to distinctly advance every branch of Mathematics then studied". His work on the subject usually referred as fluxions or calculus. Some of his contributions are given below

- He is credited with the generalized binomial theorem, which describes the algebraic expansion of power of a binomial.
- He made substantial contributions to the theory of finite differences.
- He was one of the first to use fractional exponents and coordinate geometry to derive solutions to Diophantine equations.
- He developed "Newton's method" for finding successively better approximations to the zeroes or roots of the function.
- He was the first to use infinite power series.
- He approximated partial sum of the harmonic series by Logarithms.
- He discovered Newton's identities, Newton's method, classified cubic plane curves.

HONOUR

- Later in life, Newton wrote a number of religion tracts. So, he acted as Member of Parliament.
- He became the best known master of the Royal Mint in 1699 until his death in 1727.
- He was made President of Royal Society and in 1705, became the first scientist ever to be knighted.

GOTTFRIED WILLHELM LEIBNIZ (1646-1716)

LIFE HISTORY

Gottfried Leibniz was a German Mathematician and Philosopher who occupies a prominent place in the history of Mathematics. He was born on 1 July, 1646. He became one of the most prolific inventors in the field of mechanical calculators. He also refined the binary number system which is the foundation of virtually all digital computers. The most important innovation of Leibniz in Mathematics is in the field of calculus. He developed differential and integral calculus. He wrote in several languages those are Latin, French and German. Leibniz was baptized as Leipzig and here mother's teachings influenced Leibniz's philosophical thoughts. He completed his bachelor's degree in Philosophy in December 1662. He was awarded his bachelor's degree in law on September 28, 1665. Leibniz earned his license to practice law and his doctorate in law in 6 November 1666.

Leibniz's first position was as a salaried secretary to an Alchemical Society in Nurembey. He began a program of self study in the Mathematics and Physics which pushed him to make major contribution in both subjects. Leibniz never married. He had many friends and admirers all over Europe.

CONTRIBUTION

- Leibniz was the first to employ the Mathematical notion of function to denote any of several geometric concepts derived from a curve, such as abscissa, ordinate, tangent, chord and the perpendicular.
- He was the first to see the coefficients of a system of linear equations could be arranged into an array, i.e. called a matrix, which can be manipulated to find the solution of the system.
- Leibniz also discovered boolean algebra and symbolic logic.
- Leibniz is credited along with Sir Isaac Newton, with the invention of infiniesimal calculus.
- He gave many notation which are used in present also like sign '$\int$' representing integration and 'd' used for differentials.
- The product rule of differential calculus is still called 'Leibniz's Law'.
- The theorem that tells how and when to differentiate under the integral sign is called Leibniz integral rule.
- Leibniz contributed a fair amount to the statics and dynamics. He devised a new theory of motion based on kinetic energy and potential energy.
- Those who advocate digital philosophy, a recent direction in cosmology, claim Leibniz as a precursor.
- Leibniz recasted Newton's theory as a vortex theory.
- He documented binary number system and invent a machine that could execute all four arithmetic operation.

HONOUR

- In 1985, the German Government created the Leibniz prize offering an annual award of 1.55 million euros for experimental results and 770000 euros for theoretical ones. It is the world's largest prize for scientific achievement.
- His writings are currently discussed not only for their anticipations and possible discoveries not yet recognized but as ways of advancing present knowledge.

BINOMIAL THEOREM AND LINEAR INEQUALITIES

FOR CLASS XI

Topicwise Collection of Best Subjective Problems

Very Short Answer Type Questions [1 Mark]

1. Find the number of terms in the expansion of $(1 + 2x + x^2)^{20}$.

2. Find the 11th term from the end in the expansion of $\left(2x - \frac{1}{x^2}\right)^{25}$.

3. Find the coefficient of x^n in the expansion of $(1 + x)(1 - x)^n$.

4. How many terms are free from radical signs in the expansion of $(x^{1/5} + y^{1/10})^{55}$? [HOTS]

5. Find the value of a so that the term independent of x in $\left(\sqrt{x} + \frac{a}{x^2}\right)^{10}$ is 405.

6. If the middle term in the binomial expansion of $\left(\frac{1}{x} + x\sin x\right)^{10}$ is equal to $\frac{63}{8}$, then find the value of x.

7. Solve $\frac{2x + 4}{x - 1} \geq 5$.

8. Solve $|x - 1| \leq 5, |x| \geq 2$.

9. Write the number of integral solutions of $\frac{x + 2}{x^2 + 1} > \frac{1}{2}$.

10. Solve the inequation $\left|\frac{2}{x - 4}\right| > 1, x \neq 4$.

Short Answer Type Questions [4 Marks]

11. Using binomial theorem, prove that
$(101)^{50} > (100)^{50} + (99)^{50}$

12. For what value of x is the ninth term in the expansion of $\{3^{\log_3 \sqrt{25^{x-1} + 7}} + 3^{(-1/8)\log_3 (5^{x-1} + 1)}\}^{10}$ is equal to 180?

13. If x^p occurs in the expansion of $(x^2 + 1/x)^{2n}$, prove that p's coefficient, is $\left\{\frac{(2n)!}{\left(\frac{4n - p}{3}\right)!\left(\frac{2n + p}{3}\right)!}\right\}$.

14. Find the greatest value of the term independent of x in the expansion of $\left(x\sin\alpha + \frac{\cos\alpha}{x}\right)^{10}$, where $\alpha \in R$.

15. Solve $\frac{|x| - 1}{|x| - 2} \geq 0, x \in R, x \neq \pm 2$. [HOTS]

16. Solve $-5 \leq \frac{2 - 3x}{4} \leq 9$.

Long Answer Type Questions [6 Marks]

17. If a_1, a_2, a_3, a_4 are the coefficients of four consecutive terms in the expansion of $(1 + x)^n$, then prove that
$$\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{2a_2}{a_2 + a_3}$$ [HOTS]

18. If O is the sum of odd terms and E that even terms in the expansion of $(x + a)^n$, prove that [HOTS]
(i) $O^2 - E^2 = (x^2 - a^2)^n$
(ii) $4OE = (x + a)^{2n} - (x - a)^{2n}$
(iii) $2(O^2 + E^2) = (x + a)^{2n} + (x - a)^{2n}$

19. Solve $|x - 1| + |x - 2| \geq 4$. [HOTS]

20. Solve the system of inequations
$\frac{x}{2x + 1} \geq \frac{1}{4}, \frac{6x}{4x - 1} < \frac{1}{2}$

To be Continued Page 59

APPLICATION OF DERIVATIVE

FOR CLASS XII

Topicwise Collection of Best Subjective Problems

Very Short Answer Type Questions [1 Mark]

1. Find the surface area of a sphere when its volume is changing at the same rate as its radius.
2. Find an angle θ, whose rate of increase is twice the rate of decrease of its cosine. [AI 2004]
3. If $y = \log_e x$, then find Δy when $x = 3$ and $\Delta x = 0.03$.
4. If $y = x^n$, then find the ratio of relative errors in y and x.
5. Write the angle made by the tangent to the curve $x = e^t \cos t$, $y = e^t \sin t$ at $t = \frac{\pi}{4}$ with X-axis.
6. Prove that the function $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing on R.
7. If $g(x)$ is a decreasing function of R and $f(x) = \tan^{-1}\{g(x)\}$. State whether $f(x)$ is increasing or decreasing on R.
8. If $f(x)$ attains a local minimum at $x = c$, then write the values of $f'(c)$ and $f''(c)$.
9. Find the value of a so that the sum of the squares of the roots of the equation $x^2 - (a-2)x - a + 1 = 0$ assume the least value.
10. If the relative error in measuring the radius of a circular plane is α, then find the relative error in measuring its area.

Short Answer Type Questions [4 Marks]

11. A man is moving away from a tower 41.6 m high at the rate of 2 m/s. Find the rate at which the angle of elevation of the top of tower is changing, when he is at a distance of 30 m from the front of the tower. Assume that the eye level of the man is 1.6 m from the ground. [AI 2006]
12. Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$. [HOTS]
13. Show that for $a \geq 1$, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing on R. [AI 2007]
14. If $f(x) = a \log|x| + bx^2 + x$ has extreme values at $x = -1$ and at $x = 2$, then find a and b.
15. Determine the quadratic curve $y = f(x)$, if it touches the line $y = x$ at the point $x = 1$ and passes through the point $(-1, 0)$. [HOTS]
16. Find the intervals in which $f(x) = \frac{x}{\log x}$ is increasing or decreasing. [AI 2009]

Long Answer Type Questions [6 Marks]

17. If in a ΔABC, the side c and the angle C remain constant, while the remaining elements are changed slightly, using differentials show that $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$. [HOTS]
18. Show that the angle between the tangent at any point P and the line joining P to the origin O is the same at all points on the curve $\log(x^2 + y^2) = k \tan^{-1}\left(\frac{y}{x}\right)$.
19. If a, b and c are real numbers, then find the intervals in which $f(x) = \begin{vmatrix} x + a^2 & ab & ac \\ ab & x + b^2 & bc \\ ac & bc & x + c^2 \end{vmatrix}$ is increasing or decreasing. [HOTS]
20. Show that $\sin^p\theta\cos^q\theta$ attains a maximum when $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$.

To be Continued Page 68

INTEGRATION

FOR CLASS XII

This section of magazine is aimed to increase the connectivity between reader and NCERT Textbooks. Readers are advised to solve these questions with the help of references of NCERT books given at the end of the sections.

1. $\int \frac{e^{5\log_e x} - e^{4\log_e x}}{e^{3\log_e x} - e^{2\log_e x}}\,dx$ is equal to

(a) $\frac{x^3}{3} + C$ (b) $\frac{x^2}{2} + C$

(c) $\frac{x^2}{4} + C$ (d) None of these

2. $\int (1 - x)\sqrt{x}\,dx$ is equal to

(a) $\frac{2}{3}x^{3/2} + \frac{5}{3}x^{5/3} + C$

(b) $\frac{4}{3}x^{4/3} - \frac{3}{2}x^{3/2} + C$

(c) $\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$

(d) None of the above

3. $\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\,dx$ is equal to

(a) $\cot x + \tan x + C$ (b) $-\cot x + \tan x + C$

(c) $\operatorname{cosec} x + \sec x + C$ (d) $-\cot x - \tan x + C$

4. $\int \tan^{-1}\left(\sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}\right)dx,\ 0 < x < \frac{\pi}{2}$ is equal to

(a) $\frac{x^2}{2} + C$ (b) $x + C$

(c) $\frac{x^3}{3} + C$ (d) None of these

5. $\int \{1 + 2\tan x(\tan x + \sec x)\}^{1/2}\,dx$ is equal to

(a) $\log|\sec x + \tan x| + \log|\sec x| + C$

(b) $\log|\sec x + \tan x| + \log|\tan x| + C$

(c) $\log|\sin x + \operatorname{cosec} x| + \log|\sec x| + C$

(d) None of the above

6. $\int \frac{\sin 2x}{\sin\left(x - \frac{\pi}{6}\right)\sin\left(x + \frac{\pi}{6}\right)}\,dx$ is equal to

(a) $\log\left|\cos^2 x - \frac{1}{8}\right| + C$ (b) $\log\left|\sin^2 x - \frac{1}{4}\right| + C$

(c) $\log|\tan x| + C$ (d) None of these

7. $\int \log x \frac{\sin\{1 + (\log x)^2\}}{x}\,dx$ is equal to

(a) $\frac{1}{2}\sin\{1 + (\log x)^2\} + C$

(b) $\frac{1}{2}\sin\{1 + \log x\} + C$

(c) $-\frac{1}{2}\cos\{1 + (\log x)^2\} + C$

(d) None of the above

8. $\int 5^{5^{5^x}}\,5^{5^x}\,5^x\,dx$ is equal to

(a) $\frac{5^{5^{5^x}}}{(\log_e 5)^2} + C$ (b) $\frac{5^{5^{5^x}}}{(\log_e 5)^3} + C$

(c) $\frac{5^{5^x}}{\log_e 5} + C$ (d) None of these

9. $\int \cot^n x \operatorname{cosec}^2 x\,dx,\ n \neq -1$ is equal to

(a) $\frac{1}{n+1}\cot^{n+1} x + C$

(b) $\frac{-1}{n+1}\cot^{n+1} x + C$

(c) $\frac{1}{n-1}\operatorname{cosec}^{n+1} x + C$

(d) None of the above

10. $\int \sec^{4/3} x \operatorname{cosec}^{8/3} x\, dx$ is equal to

(a) $\frac{3}{5}\tan^{1/3} x + 3\tan^{1/3} x + C$

(b) $\frac{3}{5}\tan^{-5/3} x - 3\tan^{-1/3} x + C$

(c) $-\frac{3}{5}\tan^{-5/3} x + 3\tan^{1/3} x + C$

(d) None of the above

11. $\int x \cos^3 x^2 \sin x^2\, dx$ is equal to

(a) $\frac{1}{8}\tan^4 x^2 + C$ (b) $\frac{1}{8}\sin^4 x^2 + C$

(c) $\frac{1}{8}\cos^4 x^2 + C$ (d) $-\frac{1}{8}\cos^4 x^2 + C$

12. $\int \frac{1}{x\{6(\log x)^2 + 7\log x + 2\}}\, dx$ is equal to

(a) $\log\left|\frac{2\log x + 1}{3\log x + 2}\right| + C$

(b) $\log\left|\frac{3\log x + 2}{2\log x + 1}\right| + C$

(c) $\log\left|\frac{2\log x - 1}{3\log x - 2}\right| + C$

(d) None of the above

13. $\int \frac{2\sin 2\phi - \cos\phi}{6 - \cos^2\phi - 4\sin\phi}\, d\phi$ is equal to

(a) $2\log|\sin^2\phi + 4\sin\phi - 5| - 7\tan^{-1}(\sin\phi - 2) + C$

(b) $2\log|\cos^2\phi - 4\cos\phi + 5| + 7\tan^{-1}(\cos\phi - 2) + C$

(c) $2\log|\sin^2\phi - 4\sin\phi + 5| + 7\tan^{-1}(\sin\phi - 2) + C$

(d) None of the above

14. $\int \sqrt{\frac{1+x}{x}}\, dx$ is equal to

(a) $\sqrt{x^2 + x} + \frac{1}{2}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x - x^2}\right| + C$

(b) $\sqrt{x^2 + x} - \frac{1}{2}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 - x}\right| + C$

(c) $\sqrt{x^2 + x} - \frac{1}{2}\log\left|\left(x - \frac{1}{2}\right) + \sqrt{x^2 + x}\right| + C$

(d) $\sqrt{x^2 + x} + \frac{1}{2}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x}\right| + C$

15. $\int \frac{x^3}{(x-1)(x-2)}\, dx$ is equal to

(a) $\frac{3}{2}\log|x + 1| + \frac{1}{2}\log|x - 2| + C$

(b) $\frac{2}{3}\log|x + 1| + \frac{1}{3}\log|x - 2| + C$

(c) $\frac{2}{3}\log|x - 1| + \frac{1}{3}\log|x + 2| + C$

(d) None of the above

16. $\int \frac{x^2}{(x-1)^3(x+1)}\, dx$ is equal to

(a) $\frac{1}{8}\log\left|\frac{x-1}{x+1}\right| - \frac{3}{4(x-1)} - \frac{1}{4(x-1)^2} + C$

(b) $\frac{1}{8}\log\left|\frac{x+1}{x-1}\right| + \frac{3}{4(x-1)} + \frac{1}{4(x-1)^2} + C$

(c) $\frac{1}{8}\log\left|\frac{x+1}{x-1}\right| + \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + C$

(d) None of the above

17. $\int \frac{x+1}{x(1+xe^x)^2}\, dx$ is equal to

(a) $\log(xe^x) + \log(1 + xe^x) + \frac{1}{xe^x} + C$

(b) $\log(xe^x) + \log(1 + xe^x) + xe^x + C$

(c) $\log(1 + e^x) + \log(xe^x) - \frac{1}{xe^x} + C$

(d) $\log(xe^x) - \log(1 + xe^x) + \frac{1}{1 + xe^x} + C$

18. $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\, dx$ is equal to

(a) $x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + 3\tan^{-1}\left(\frac{x}{3}\right) + C$

(b) $x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3\tan^{-1}\left(\frac{x}{2}\right) + C$

(c) $x + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right) - 3\tan^{-1}\left(\frac{2x}{3}\right) + C$

(d) None of the above

19. $\int \frac{\log(1+x^2)}{x^3}\, dx$ is equal to

(a) $-\frac{1}{2x^2}\log(1 + x^2) + \frac{1}{2}\log\left|\frac{x^2}{x^2+1}\right| + C$

(b) $\frac{1}{2x^2}\log(1 + x^2) - \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right| + C$

(c) $-\frac{1}{2x^2}\log(1 + x^2) - \frac{1}{2}\log\left|\frac{1-x^2}{x^2}\right| + C$

(d) None of the above

20. $\int \frac{e^x(1-x)^2}{(1+x^2)^2}\, dx$ is equal to

(a) $-\frac{e^x}{1+x^2} + C$ (b) $\frac{1+x^2}{e^x} + C$ (c) $\frac{e^x}{1-x^2}$ (d) $\frac{e^x}{1+x^2} + C$

21. If $I_1 = \int e^{ax}\cos bx\, dx$ and $I_2 = \int e^{ax}\sin bx\, dx$, then the value of $\tan^{-1}\left(\frac{I_2}{I_1}\right) + \tan^{-1}\left(\frac{b}{a}\right)$ is

(a) bx^2 (b) ax (c) bx (d) $\frac{a}{x}$

22. $\int_0^1 x \log(1+2x)\,dx$ is equal to

(a) $\frac{3}{8}\log 3$ (b) $\frac{3}{8}\log 5$ (c) $\frac{2}{5}\log 8$ (d) $\frac{5}{3}\log 2$

23. $\int_0^{\pi/6} (2+3x^2)\cos 3x\,dx$ is equal to

(a) $\frac{1}{18}(\pi^2-16)$ (b) $\frac{1}{36}(\pi^2-16)$

(c) $\frac{1}{36}(\pi^2+16)$ (d) None of these

24. $\int_{\pi/6}^{\pi/3} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$ is equal to

(a) $2\sin^{-1}\left(\frac{\sqrt{3}+1}{2}\right)$ (b) $\frac{1}{2}\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$

(c) $\sin^{-1}(\sqrt{3}+1)$ (d) $2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$

25. $\int_0^{\pi/2} \sin 2x \log(\tan x)\,dx$ is equal to

(a) 1 (b) 2 (c) 0 (d) $\frac{1}{2}$

26. $\int_{\pi/2}^{\pi/2} |\sin x|\,dx$ is equal to

(a) 2 (b) – 2 (c) – 1 (d) 0

27. $\int_{\pi/4}^{\pi/4} \frac{\sec^2 x}{1+e^x}\,dx$ is equal to

(a) 1 (b) 2 (c) 3 (d) – 1

28. $\int_0^{\pi} \frac{x}{1-\cos\alpha\sin x}\,dx$ is equal to

(a) $\frac{\pi(\pi+\alpha)}{\cos\alpha}$

(b) $\frac{\pi(\pi-\alpha)}{\cos\alpha}$

(c) $\frac{\pi(\pi-\alpha)}{\sin\alpha}$

(d) None of the above

29. If $[\cdot]$ denotes the greatest integer function, then the value of $\int_1^2 [3x]\,dx$

(a) 2 (b) – 2 (c) 4 (d) 1

30. $\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}\,dx$ is equal to

(a) π (b) π^2 (c) $\frac{\pi^2}{2}$ (d) $\frac{\pi^2}{4}$

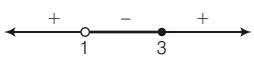
Correct Answers along with NCERT Textbook Reference

1.	**(a)** Page no. 290	Important Formulae
2.	**(c)** Page no. 290	Important Formulae
3.	**(d)** Page no. 290	Important Formulae
4.	**(a)** Page no. 290	Important Formulae
5.	**(a)** Page no. 300	Integration by Substitution
6.	**(b)** Page no. 300	Integration by Substitution
7.	**(c)** Page no. 300	Integration by Substitution
8.	**(b)** Page no. 300	Integration by Substitution
9.	**(b)** Page no. 305	Integration using Trigonometric Identities
10.	**(c)** Page no. 305	Integration using Trigonometric Identities
11.	**(d)** Page no. 305	Integration using Trigonometric Identities
12.	**(a)** Page no. 307	Integration of Some Particular Functions
13.	**(c)** Page no. 307	Integration of Some Particular Functions
14.	**(d)** Page no. 307	Integration of Some Particular Functions
15.	**(d)** Page no. 316	Integration by Partial Fractions
16.	**(a)** Page no. 316	Integration by Partial Fractions
17.	**(d)** Page no. 316	Integration by Partial Fractions
18.	**(b)** Page no. 316	Integration by Partial Fractions
19.	**(a)** Page no. 323	Integration by Parts
20.	**(d)** Page no. 323	Integration by Parts
21.	**(c)** Page no. 323	Integration by Parts
22.	**(a)** Page no. 331	Definite Integral
23.	**(c)** Page no. 331	Definite Integral
24.	**(d)** Page no. 331	Definite Integral
25.	**(c)** Page no. 341	Properties of Definite Integrals
26.	**(a)** Page no. 341	Properties of Definite Integrals
27.	**(a)** Page no. 341	Properties of Definite Integrals
28.	**(c)** Page no. 341	Properties of Definite Integrals
29.	**(d)** Page no. 341	Properties of Definite Integrals
30.	**(b)** Page no. 341	Properties of Definite Integrals

Continued from Page 55

ANSWERS

1. 41
2. $-{}^{25}C_{15}\times\frac{2^{10}}{x^{20}}$
3. $(-1)^n(1-n)$
4. 6
5. ± 3
6. $n\pi+(-1)^n\frac{\pi}{6}$, $n\in Z$
7. $1<x\le 3$



8. $x\in[-4,-2]\cup[2,6]$
9. $x\in\{0,1,2\}$
10. $x\in(2,6)-\{4\}$
12. $x=\log_5 15$ or 1
14. $\frac{10!}{2^5(5!)^2}$
15. $x\in[-1,1]\cup(-\infty,-2)\cup(2,\infty)$
16. $x\in\left[\frac{-34}{3},\frac{22}{3}\right]$
19. $x\in(-\infty,-1/2]\cup[7/2,\infty)$
20. $x\in(-1/8,1/4)$

Best Integer Type Problems On
DIFFERENTIATION

A collection of chapterwise best problems of their types

1. Let f be a twice differentiable function such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h'(x) = [f(x)]^2 + [g(x)]^2$, $h(1) = 8$ and $h(0) = 2$, then $h(2) - 7$ is equal to

2. If $f(x) = \sin\left\{\frac{\pi}{3}[x] - x^2\right\}$ for $2 < x < 3$ and $[x]$ denotes the greatest integer less than or equal to x, then $f'\left(\sqrt{\frac{\pi}{3}}\right) + \sqrt{\frac{\pi}{3}}$ is equal to

3. If $f'(x) = \phi(x)$ and $\phi'(x) = f(x)$ for all x. Also, $f(3) = 5$ and $f'(3) = 4$. Then, the value of $[\{f(10)\}^2 - \{\phi(10)\}^2]$ is

4. Let $f(x) = x^2 + xg'(1) + g''(2)$ and $g(x) = x^2 + xf'(2) + f''(3)$. Then, the value of $g''(2) + f''(3)$ is

5. Let f be a real function such that $f(x - y)$, $f(x)f(y)$ and $f(x + y)$ are in AP for all $x, y \in R$. If $f(0) \neq 0$, then the value of $f'(3) + f'(-3)$ is

6. Let $f(x) = 3x^2 + 4xg'(1) + g''(2)$ and $g(x) = 2x^2 + 3xf'(2) + f''(3)$ for all $x \in R$, if $f''(3) + g''(2) + \lambda = 12$, then the value of λ is

7. If a curve is represented parametrically by the equations $x = 4t^3 + 3$ and $y = 4 + 3t^4$ and $\frac{d^2x}{dy^2} \Big/ \left(\frac{dx}{dy}\right)^n$ is constant, then the value of n is

8. If $f(x) = \cos^2 x + \cos^2\left(x + \frac{\pi}{3}\right) + \sin x \sin\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 3$, then $\frac{d}{dx}\{gof(x)\}$ is equal to

9. If $f(x) = \begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\frac{n\pi}{2} & \cos\frac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$, then the value of $\frac{d^n}{dx^n}[f(x)]$ at $x = 0$ for $n = 2x + 1$ is

10. If $f(x) = x^n$, n being a non-negative integer, then the number of values of n for which $f'(\alpha + \beta) = f'(\alpha) + f'(\beta)$ for all $\alpha, \beta > 0$, are

11. If $f(x) = \frac{x^2 - x}{x^2 + 2x}$, then $\frac{d}{dx}\{f^{-1}(x)\} - \frac{3}{(1-x)^2}$ is equal to

12. Let $f(x)$ be a second degree polynomial function such that $f(-1) = f(1)$ and α, β, γ are in AP. Then, $\frac{f'(\alpha) + f'(\gamma)}{f'(\beta)}$ is

13. If $y = \sin px$ and y_n is nth derivative of y, then $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ is equal to

14. Let $f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$ for all real x and y, if $f'(0)$ exists and equals -1 and $f(0) = 1$, then find the value of $f(2) + 2$.

15. $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ for all $x, y \in R$, $(xy \neq 1)$ and $\lim_{x \to 0} \frac{f(x)}{x} = 2$, then the value of $f'(1) + 5$ is

16. If the function $f(x) = -4^{\frac{1-x}{2}} + 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$ and $g(x) = f^{-1}(x)$, then the reciprocal of $g'\left(\frac{7}{6}\right)$ is

17. Let $g(x) = \begin{cases} \frac{x^2 + x\tan x - x\tan 2x}{ax + \tan x - \tan 3x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

If $g'(0)$ exists and is equal to non-zero value b, then $52\frac{b}{a}$ is equal to

18. A function is represented parametrically by the equations $x = \frac{1+t}{t^3}$, $y = \frac{3}{2t^2} + \frac{2}{t}$. Then, the value of $\left| \frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^3 \right|$ is

19. If $y = \frac{2^{\log_{2^{1/4}} x} - 3^{\log_{27}(x^2+1)^3} - 2x}{7^{4\log_{49} x} - x - 1}$ and $\frac{dy}{dx} = ax + b$, then the value of $a + b$ is

20. If $\lim_{t \to x} \frac{e^t f(x) - e^x f(t)}{(t-x)\{f(x)\}^2} = 2$ and $f(0) = \frac{1}{2}$, then the value of $f'(0)$ is

21. If $f(x) = e^x - e^{-x} - 2\sin x - \frac{2}{3}x^3$. Then, the least value of n for which $\frac{d^n}{dx^n} f(x)\Big|_{x=0}$ is non-zero, is

22. Suppose $f(x) = e^{ax} + e^{bx}$, where $a \neq b$ and that $f''(x) = 2f'(x) - 15f(x) = 0$ for all x. Then, the value of $\frac{|ab|}{3}$ is

23. $y = f(x)$, where f satisfies the relation $f(x + y) = 2f(x) + xf(y) + y\sqrt{f(x)}$, $\forall\ x, y \in R$ and $f(0) = 0$, then $f(6)$ is equal to

24. A function $f : R \to R$ satisfies $\sin x \cos y \{f(2x + 2y) - f(2x - 2y)\} = \cos x \sin y \{f(2x + 2y) + f(2x - 2y)\}$. If $f'(0) = \frac{1}{2}$, then $4f''(x) + f(x)$ is equal to

25. Let $u(x) + v(x)$ be differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to

26. If $g(x + y) = g(x) + g(y) + 3xy(x + y)$, $\forall\ x,\ y \in R$ and $g'(0) = -4$, then number of real roots of the equation $g(x) = 0$ is

27. A curve is represented parametrically by the equations $x = f(t) = a^{\ln(b^t)}$ and $y = g(t) = b^{-\ln(a^t)}$ $a, b > 0$ and $a \neq 1,\ b \neq 1$, where $t \in R$, then the value of $\frac{d^2y}{dx^2}$ at the point where $f(t) = g(t)$ is

28. If $e^x = \frac{\sqrt{1+t} - \sqrt{1-t}}{\sqrt{1+t} + \sqrt{1-t}}$, $\tan\frac{y}{2} = \sqrt{\frac{1-t}{1+t}}$ and $\frac{dy}{dx}$ at $t = \frac{1}{2}$ is $-\frac{a}{2}$, then the value of a is

29. If $(\sin y)^{\sin\left(\frac{\pi}{2}x\right)} + \frac{\sqrt{3}}{2}\sec^{-1}(2x) + 2^x \tan[\log(x + 2)] = 0$, then find $\frac{dy}{dx}$ at $x = -1$.

30. If $z = (\cos x)^5$ and $y = \sin x$. Then, the value of $2\frac{d^2z}{dy^2}$ at $x = \frac{2\pi}{9}$ is

Answers with Explanation

1. ***(7)*** We have, $h'(x) = [f(x)]^2 + [g(x)]^2$

$\Rightarrow$ $h''(x) = 2f(x)f'(x) + 2g(x)g'(x)$

$\Rightarrow$ $h''(x) = 2f(x)g(x) + 2g(x)f''(x)$ $[\because g(x) = f'(x) \therefore g'(x) = f''(x)]$

$\Rightarrow$ $h''(x) = 2f(x)g(x) + 2g(x)\{-f(x)\}$ $[\because f''(x) = -f(x)]$

$\Rightarrow$ $h''(x) = 0$

$\Rightarrow$ $h'(x) = C$ and $h(1) = C + C_1$

$\Rightarrow$ $2 = C_1$ and $8 = C + C_1$

$\Rightarrow$ $C_1 = 2$ and $C = 6$

$\therefore$ $h(x) = 6x + 2 \Rightarrow h(2) = 6 \times 2 + 2 = 14$

Hence, $h(2) - 7 = 14 - 7 = 7$

2. ***(0)*** For $2 < x < 3$

We have, $[x] = 2$

$\therefore$ $f(x) = \sin\left(\frac{2\pi}{3} - x^2\right)$

$\Rightarrow$ $f'(x) = -2x\cos\left(\frac{2\pi}{3} - x^2\right)$

$\Rightarrow$ $f'\left(\sqrt{\frac{\pi}{3}}\right) = -2\sqrt{\frac{\pi}{3}}\cos\frac{\pi}{3} = -\sqrt{\frac{\pi}{3}}$

$\therefore$ $f'\left(\sqrt{\frac{\pi}{3}}\right) + \sqrt{\frac{\pi}{3}} = -\sqrt{\frac{\pi}{3}} + \sqrt{\frac{\pi}{3}} = 0$

3. ***(9)*** We have, $\frac{d}{dx}[\{f(x)\}^2 - \{\phi(x)\}^2]$

$= 2f(x)f'(x) - 2\phi(x)\phi'(x)$

$= 2f(x)\phi(x) - 2\phi(x)f(x)$

$= 0$ $[\because f' = \phi$ and $\phi' = f]$

$\therefore$ $\{f(x)\}^2 - \{\phi(x)\}^2 = C$ (constant), $\forall\ x$ ...(i)

$\Rightarrow$ $\{f(3)\}^2 - \{\phi(3)\}^2 = C$

$\Rightarrow$ $5^2 - 4^2 = C$ $[\because f(3) = 5$ and $\phi(3) = f'(3) = 4]$

$\Rightarrow$ $C = 9$

On substituting the value of C in Eq. (i), we get

$\{f(x)\}^2 - \{\phi(x)\}^2 = 9, \forall\ x$

$\Rightarrow$ $\{f(10)\}^2 - \{\phi(10)\}^2 = 9$

4. ***(4)*** We have,

$$f(x) = x^2 + xg'(1) + g''(2) \text{ and } g(x) = x^2 + xf'(2) + f''(3)$$

$\Rightarrow$ $f'(x) = 2x + g'(1)$ and $g'(x) = 2x + f'(2)$ ...(i)

On differentiating again Eq. (i) w.r.t. x, we get

$$f''(x) = 2 \quad \text{and} \quad g''(x) = 2$$

$\Rightarrow$ $f''(3) = 2$ and $g''(2) = 2$

Hence, $g''(2) + f''(3) = 2 + 2 = 4$

5. ***(0)*** It is given that $f(x - y)$, $f(x)\, f(y)$ and $f(x + y)$ are in AP, $\forall\ x, y \in R$.

$\therefore$ $2f(x)f(y) = f(x - y) + f(x + y), \forall\ x, y \in R$ ...(i)

Putting $x = y = 0$, we get $2\{f(0)\}^2 = 2f(0)$

$\Rightarrow$ $f(0)\{f(0) - 1\} = 0$ $\Rightarrow$ $f(0) = 1$ $[\because f(0) \neq 0]$

Putting $x = 0$ and $y = x$ in Eq. (i), we get

$$2f(0)f(x) = f(-x) + f(x), \forall\ x \in R$$

$\Rightarrow$ $2f(x) = f(-x) + f(x), \forall\ x \in R$

$\Rightarrow$ $f(-x) = f(x), \forall\ x \in R$

$\Rightarrow$ $f(x)$ is an even function.

$\Rightarrow$ $f'(x)$ is an odd function.

$\therefore$ $f(-1) = f(1)$ and $f'(-3) = -3f'(3)$

$\Rightarrow$ $f(1) = f(-1)$ and $f'(3) + f'(-3) = 0$

6. ***(2)*** We have, $f(x) = 3x^2 + 4xg'(1) + g''(2)$ ...(i)

and $g(x) = 2x^2 + 3xf'(2) + f''(3)$ ...(ii)

$\therefore$ $f'(x) = 6x + 4g'(1)$ ...(iii)

and $g'(x) = 4x + 3f'(2)$ ...(iv)

$\Rightarrow$ $f''(x) = 6$ ...(v)

and $g''(x) = 4$ ...(vi)

Putting $x = 1$ in Eqs. (iii) and (iv), we get

$$f'(1) = 6 + 4g'(1) \quad \text{and} \quad g'(1) = 4 + 3f'(2)$$

$\Rightarrow$ $f'(1) = 6 + 4\{4 + 3f'(2)\}$ $\Rightarrow$ $f'(1) = 22 + 12f'(2)$

Putting $x = 2$ in Eq. (iv), we get $g'(2) = 8 + 3f'(2)$

$\Rightarrow$ $g'(2) = 8 + 3\{12 + 4g'(1)\}$ $\Rightarrow$ $g'(2) = 44 + 12g'(1)$

Putting $x = 3$ in Eq. (v) and $x = 2$ in Eq. (vi), we get

$$f''(3) = 6 \quad \text{and} \quad g''(2) = 4$$

$\Rightarrow$ $f''(3) + g''(2) = 6 + 4 = 10$

$\because$ $f''(3) + g''(2) + \lambda = 12$

$\therefore$ $10 + \lambda = 12$ $\Rightarrow$ $\lambda = 2$

7. ***(5)*** We have, $x = 4t^3 + 3$ and $y = 4 + 3t^4$

$$\frac{dx}{dt} = 12t^2 \text{ and } \frac{dy}{dt} = 12t^3$$

$\therefore$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{12t^3}{12t^2} = t$ $\Rightarrow$ $\dfrac{dx}{dy} = \dfrac{1}{t} = t^{-1}$

$\Rightarrow$ $\dfrac{d^2x}{dy^2} = -t^{-2}\dfrac{dt}{dy} = -t^{-2} \times \dfrac{1}{12t^3} = -\dfrac{1}{12}t^{-5}$

$\therefore$ $\dfrac{d^2x}{dy^2} \Big/ \left(\dfrac{dx}{dy}\right)^n = -\dfrac{1}{12t^5} \times t^n = -\dfrac{1}{12}t^{n-5}$

It is given that $\dfrac{d^2x}{dy^2} \Big/ \left(\dfrac{dx}{dy}\right)^n$ is constant.

Therefore, $n = 5$

8. ***(0)*** We have, $f(x) = \cos^2 x + \cos^2\left(x + \dfrac{\pi}{3}\right) + \sin x \sin\left(x + \dfrac{\pi}{3}\right)$

$\Rightarrow$ $f(x) = \dfrac{1}{2}\left[1 + \cos 2x + 1 + \cos\left(2x + \dfrac{2\pi}{3}\right) + \cos\dfrac{\pi}{3} - \cos\left(2x + \dfrac{\pi}{3}\right)\right]$

$\Rightarrow$ $f(x) = \dfrac{1}{2}\left[\dfrac{5}{2} + \cos 2x + \cos\left(2x + \dfrac{2\pi}{3}\right) - \cos\left(2x + \dfrac{\pi}{3}\right)\right]$

$\Rightarrow$ $f(x) = \dfrac{1}{2}\left[\dfrac{5}{2} + 2\cos\left(2x + \dfrac{\pi}{3}\right)\cos\dfrac{\pi}{3} - \cos\left(2x + \dfrac{\pi}{3}\right)\right]$

$\Rightarrow$ $f(x) = \dfrac{5}{4}$

$\therefore$ $gof(x) = g\left(\dfrac{5}{4}\right) = 3, \forall\ x$ $\Rightarrow$ $\dfrac{d}{dx}\{gof(x)\} = 0, \forall\ x$

9. ***(0)*** We have, $f(x) = \begin{vmatrix} x^n & \sin x & \cos x \\ n! & \sin\dfrac{n\pi}{2} & \cos\dfrac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$

$\Rightarrow$ $\dfrac{d^n}{dx^n}f(x) = \begin{vmatrix} n! & \sin\left(\dfrac{n\pi}{2} + x\right) & \cos\left(\dfrac{n\pi}{2} + x\right) \\ n! & \sin\dfrac{n\pi}{2} & \cos\dfrac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix}$

$\Rightarrow$ $\left[\dfrac{d^n}{dx^n}\{f(x)\}\right]_{x=0} = \begin{vmatrix} n! & \sin\dfrac{n\pi}{2} & \cos\dfrac{n\pi}{2} \\ n! & \sin\dfrac{n\pi}{2} & \cos\dfrac{n\pi}{2} \\ a & a^2 & a^3 \end{vmatrix} = 0$

10. ***(2)*** We have, $f(x) = x^n$ $\Rightarrow$ $f'(x) = nx^{n-1}$

$\therefore$ $f'(\alpha + \beta) = f'(\alpha) + f'(\beta), \forall\ \alpha, \beta > 0$

$\Rightarrow$ $n(\alpha + \beta)^{n-1} = n\alpha^{n-1} + n\beta^{n-1}, \forall\ \alpha, \beta > 0$

$\Rightarrow$ $(\alpha + \beta)^{n-1} = \alpha^{n-1} + \beta^{n-1}, \forall\ \alpha, \beta > 0$

$\Rightarrow$ $n - 1 = 1$ $\Rightarrow$ $n = 2$

Also, for $n = 0$, we have $f(x) = 1, \forall\ x$ $\Rightarrow$ $f'(x) = 0, \forall\ x$

$\therefore$ $f'(\alpha + \beta) = f'(\alpha) + f'(\beta)$ $\Rightarrow$ $n = 0, 2$

Hence, number of values of n is 2.

11. ***(0)*** We have, $f(x) = \dfrac{x^2 - x}{x^2 + 2x}$

Clearly, $f(x)$ is not defined at $x = 0, -2$.

So, domain $(f) = R - \{-2, 0\}$, $\forall\ x \in$ Domain (f), we have

$$f(x) = \frac{x^2 - x}{x^2 + 2x} = \frac{x - 1}{x + 2} = 1 - \frac{3}{x + 2}$$

Now, $fof^{-1}(x) = x$ $\Rightarrow$ $f\{f^{-1}(x)\} = x$

$\Rightarrow$ $1 - \dfrac{3}{f^{-1}(x) + 2} = x$ $\Rightarrow$ $1 - x = \dfrac{3}{f^{-1}(x) + 2}$

$\Rightarrow$ $f^{-1}(x) = \dfrac{3}{1 - x} - 2$ $\Rightarrow$ $\dfrac{d}{dx}\{f^{-1}(x)\} = \dfrac{3}{(1 - x)^2}$

$\therefore$ $\dfrac{d}{dx}\{f^{-1}(x)\} - \dfrac{3}{(1 - x)^2} = \dfrac{3}{(1 - x)^2} - \dfrac{3}{(1 - x)^2} = 0$

12. ***(2)*** Let $f(x) = ax^2 + bx + c$. Then, $f'(x) = 2ax + b$

Now, $f(-1) = f(1) \Rightarrow a - b + c = a + b + c \Rightarrow b = 0$

$\therefore$ $f'(x) = 2ax \Rightarrow f'(\alpha) = 2a\alpha, f'(\beta) = 2a\beta$

and $f'(\gamma) = 2a\gamma$

Since, α, β, γ are in AP. Therefore, $2\beta = \alpha + \gamma$

$\Rightarrow$ $4a\beta = 2a\alpha + 2a\gamma \Rightarrow 2f'(\beta) = f'(\alpha) + f'(\gamma)$

$\Rightarrow$ $\dfrac{f'(\alpha) + f'(\gamma)}{f'(\beta)} = 2$

13. ***(0)*** $D = \begin{vmatrix} \sin px & p\cos px & -p^2 \sin px \\ -p^3 \cos px & p^4 \sin px & p^5 \cos px \\ -p^6 \sin px & -p^7 \cos px & p^8 \sin px \end{vmatrix}$

$= p^9 \begin{vmatrix} \sin px & p\cos px & -p^2 \sin px \\ -\cos px & p\sin px & p^2 \cos px \\ -\sin px & -p\cos px & p^2 \sin px \end{vmatrix}$

$= -p^9 \begin{vmatrix} \sin px & p\cos px & -p^2 \sin px \\ -\cos px & p\sin px & p^2 \cos px \\ \sin px & p\cos px & -p^2 \sin px \end{vmatrix} = 0$

14. ***(1)*** Since, $f\left(\dfrac{x+y}{2}\right) = \dfrac{f(x) + f(y)}{2}$ replacing x by $2x$ and y by 0, we get

$$f(x) = \frac{f(2x) + f(0)}{2} \Rightarrow f(2x) + f(0) = 2f(x)$$

$\Rightarrow$ $f(2x) - 2f(x) = -f(0)$ ...(i)

Now, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{f\left(\frac{2x+2h}{2}\right) - f(x)}{h}$

$= \lim\limits_{h \to 0} \left\{ \dfrac{\frac{f(2x) + f(2h)}{2} - f(x)}{h} \right\}$

$= \lim\limits_{h \to 0} \left\{ \dfrac{f(2x) + f(2h) - 2f(x)}{2h} \right\}$

$= \lim\limits_{h \to 0} \left\{ \dfrac{f(2h) - f(0)}{2h} \right\}$ [from Eq. (i)]

$= f'(0) = -1, \forall\ x \in R$ [given]

Integrating, we get $f(x) = -x + C$

Putting $x = 0$, we get

$f(0) = 0 + C = 1$ [given]

$\therefore$ $C = 1$

Then, $f(x) = 1 - x$

$\therefore$ $f(2) = 1 - 2 = -1$

Hence, $f(2) + 2 = -1 + 2 = 1$

15. ***(6)*** Given, $f(x) + f(y) = f\left(\dfrac{x+y}{1-xy}\right)$ ...(i)

Putting $x = y = 0$, we get $f(0) = 0$

Putting $y = -x$, we get $f(x) + f(-x) = f(0)$

$\Rightarrow$ $f(-x) = -f(x)$ ...(ii)

Also, $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = 2$

Now, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ ...(iii)

$= \lim\limits_{h \to 0} \dfrac{f(x+h) + f(-x)}{h}$ [using Eq. (ii)]

$= \lim\limits_{h \to 0} \dfrac{f\left(\frac{x+h-x}{1-(x+h)(-x)}\right)}{h}$ [using Eq. (i)]

$= \lim\limits_{h \to 0} \left[\dfrac{f\left(\frac{h}{1+x(x+h)}\right)}{h} \right]$

$= \lim\limits_{h \to 0} \dfrac{f\left(\frac{h}{1+xh+x^2}\right)}{\frac{h}{1+xh+x^2}} \times \left(\dfrac{1}{1+xh+x^2}\right)$

$= \lim\limits_{h \to 0} \dfrac{f\left(\frac{h}{1+xh+x^2}\right)}{\frac{h}{1+xh+x^2}} \times \lim\limits_{h \to 0} \dfrac{1}{1+xh+x^2}$

$\left[\text{using } \lim\limits_{x \to 0} \dfrac{f(x)}{x} = 2\right]$

$= 2 \times \dfrac{1}{1+x^2} = \dfrac{2}{1+x^2}$

On integrating both sides, we get

$f(x) = 2\tan^{-1} x + C$ where $f(0) = 0 \Rightarrow C = 0$

Thus, $f(x) = 2\tan^{-1} x$ and $f'(1) = \dfrac{2}{1+1^2} = \dfrac{2}{2} = 1$

$\therefore$ $f'(1) + 5 = 1 + 5 = 6$

16. ***(5)*** We have, $gof(x) = x$

or $g'\{f(x)\}\ f'(x) = 1$

When $f(x) = -\dfrac{7}{6}$, $x = 1$

or $g'\{f(x)\}\ g'\left(\dfrac{7}{6}\right) f'(1) = 1$

Hence, $g'\left(\dfrac{7}{6}\right) = \dfrac{1}{f'(1)} = \dfrac{1}{5} \Rightarrow \dfrac{1}{g'\left(\frac{7}{6}\right)} = 5$

17. ***(7)*** $g'(0) = b = \lim\limits_{x \to 0} \dfrac{x^2 + x\tan x - x\tan 2x}{x(ax + \tan x - \tan 3x)}$

$= \lim\limits_{x \to 0} \dfrac{x + \tan x - \tan 2x}{ax + \tan x - \tan 3x}$

$= \lim\limits_{x \to 0} \dfrac{x + \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots \infty\right) - \left(2x + \frac{8x^3}{3} + \frac{2}{15} \times 32x^5 + \ldots \infty\right)}{ax + \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots \infty\right) - \left(3x + \frac{27}{3}x^3 + \frac{2}{15} \times 243x^5 + \ldots \infty\right)}$

$$= \lim_{x \to 0} \frac{x^3\left(\frac{7}{3} + \left(\frac{62}{15}\right)x^2 + \ldots \infty\right)}{(a+1-3)x + \left(-9\frac{1}{3}\right)x^3 + \frac{2}{15}(-242)x^5 + \ldots \infty}$$

b can be finite, if $a + 1 - 3 = 0$.

Therefore, $a = 2$

and $\quad b = \dfrac{-\frac{7}{3}}{\frac{1}{3} - 9} = \left(\frac{7}{3}\right)\left(\frac{3}{26}\right) = \frac{7}{26}$

$\Rightarrow \quad 52\dfrac{b}{a} = 7$

18. ***(1)*** $\dfrac{dx}{dt} = \dfrac{-3}{t^4} - \dfrac{2}{t^3} = -\left(\dfrac{3 + 2t}{t^4}\right)$

$$\frac{dy}{dt} = -\left(\frac{3}{t^3}\right) + \frac{2}{t^2} = -\left(\frac{3 + 2t}{t^3}\right)$$

$\Rightarrow \quad \dfrac{dy}{dx} = t \Rightarrow \dfrac{dy}{dx} - x\left(\dfrac{dy}{dx}\right)^3 = t - \left(\dfrac{1 + t}{t^3}\right)t^3 = -1$

Hence, $\quad \left|\dfrac{dy}{dx} - x\left(\dfrac{dy}{dx}\right)^3\right| = 1$

19. ***(3)*** Given, $\quad y = \dfrac{2^{\log_{2^{1/4}} x} - 3^{\log_{27}(x^2+1)^3} - 2x}{7^{4\log_{49} x} - x - 1}$

$$= \frac{x^4 - (x^2 + 1) - 2x}{x^2 - x - 1}$$

$$= \frac{x^4 - (x^2 + 1 + 2x)}{x^2 - x - 1} = x^2 + x + 1$$

$\therefore \quad \dfrac{dy}{dx} = 2x + 1 = ax + b$

Hence, $\quad a = 2 \quad$ and $\quad b = 1$

$\therefore \quad a + b = 2 + 1 = 3$

20. ***(0)*** $\lim\limits_{t \to x} \dfrac{e^t f(x) - e^x f(t)}{(t - x)\{f(x)\}^2} = 2$

$\Rightarrow \quad \lim\limits_{t \to x} \dfrac{e^t f(x) - e^x f'(t)}{1 \times \{f(x)\}^2} = 2$

$\Rightarrow \quad \dfrac{e^x f(x) - e^x f'(x)}{1 \times \{f(x)\}^2} = 2 \Rightarrow \dfrac{d}{dx}\left(\dfrac{e^x}{f(x)}\right) = 2$

$\Rightarrow \quad \dfrac{e^x}{f(x)} = 2x + C \qquad$ [on integrating]

$f(0) = \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{1}{f(0)} = C$

$C = 2$

$\Rightarrow \quad f(x) = \dfrac{e^x}{2x + 2}$

$f'(x) = \dfrac{e^x(2x + 2) - 2e^x}{(2x + 2)^2}$

$\Rightarrow \quad f'(0) = \dfrac{e^0(2 \times 0 + 2) - 2e^0}{(2 \times 0 + 2)^2} = \dfrac{0}{4} = 0$

21. ***(7)*** Given, $f(x) = e^x - e^{-x} - 2\sin x - \dfrac{2}{3}x^3$

$f'(x) = e^x + e^{-x} - 2\cos x - 2x^2$

$f''(x) = e^x - e^{-x} + 2\sin x - 4x$

$f'''(x) = e^x + e^{-x} + 2\cos x - 4$

$f^{IV}(x) = e^x - e^{-x} - 2\sin x$

$f^{V}(x) = e^x + e^{-x} - 2\cos x$

$f^{VI}(x) = e^x - e^{-x} + 2\sin x$

$f^{VII}(x) = e^x + e^{-x} + 2\cos x$

Clearly, $f^{VII}(0)$ is non-zero.

Hence, the least value of n is 7.

22. ***(5)*** According to question,

$(a^2 - 2a - 15)e^{ax} + (b^2 - 2b - 15)e^{bx} = 0$

$\Rightarrow \quad a^2 - 2a - 15 = 0$ and $b^2 - 2b - 15 = 0$

$\Rightarrow \quad (a - 5)(a + 3) = 0$ and $(b - 5)(b + 3) = 0$

i.e. $\quad a = 5$ or $a = -3$ and $b = 5$ or $b = -3$

$\therefore \quad a \neq b$

Hence, $\quad a = 5 \quad$ and $\quad b = -3$

$\Rightarrow \quad a = -3 \quad$ and $\quad b = 5$

$\Rightarrow \quad ab = -15$

$\therefore \quad |ab|/3 = 15/3 = 5$

23. ***(9)*** $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x + 0)}{h}$

$\Rightarrow \quad \lim\limits_{h \to 0} \dfrac{2f(x) + xf(h) + h\sqrt{f(x)} - 2f(x) - xf(0) - 0\sqrt{f(x)}}{h}$

$\Rightarrow \quad f(0) = 0$

$\Rightarrow \quad \lim\limits_{h \to 0} x\left(\dfrac{f(h) - f(0)}{h - 0}\right) + \sqrt{f(x)} = f'(0) + \sqrt{f(x)}$ [$\because f'(0) = 0$]

$\Rightarrow \quad f'(x) = \sqrt{f(x)} \Rightarrow \int \dfrac{f'(x)}{\sqrt{f(x)}}\,dx = \int dx$

$\Rightarrow \quad 2\sqrt{f(x)} = x + C \Rightarrow f(x) = \dfrac{x^2}{4}$ [$\because f(0) = 0$]

$\therefore \quad f(6) = \dfrac{(6)^2}{4} = 9$

24. ***(0)*** We have, $\quad \dfrac{f(2x + 2y)}{f(2x - 2y)} = \dfrac{\sin(x + y)}{\sin(x - y)}$

$\Rightarrow \quad \dfrac{f(\alpha)}{\sin\frac{\alpha}{2}} = \dfrac{f(\beta)}{\sin\frac{\beta}{2}} = k \quad$ or $\quad f(x) = k\sin\dfrac{x}{2}$

$\therefore \quad f'(x) = \dfrac{k}{2}\cos\dfrac{x}{2}$

and $\quad f''(x) = \dfrac{-k}{4}\sin\dfrac{x}{2}$

$\Rightarrow \quad 4f''(x) + f(x) = 0$

25. ***(1)*** $u(x) = 7v(x) \Rightarrow u'(x) = 7v'(x)$

or $\quad p = 7 \qquad$ [given]

Again, $\quad \dfrac{u(x)}{v(x)} = 7 \quad$ or $\quad \left(\dfrac{u(x)}{v(x)}\right)' = 0$ or $q = 0$

Now, $\quad \dfrac{p + q}{p - q} = \dfrac{7 + 0}{7 - 0} = 1$

26. ***(3)*** $g(x+y)=g(x)+g(y)+3x^2y+3xy^2$ …(i)

or $g'(x+y)=g'(x)+6yx+3y^2$

[differentiating w.r.t. x keeping y as constant]

Putting $x=0$. Then, $g'(y)=g'(0)+3y^2=-4+3y^2$

$\therefore\quad g'(x)=-4+3x^2 \quad\therefore\quad g(x)=-4x+x^3+C$

Now, putting $x=y=0$ in Eq. (i).

Then, $g(0)=g(0)+g(0)+0$

or $g(0)=0 \quad\therefore\quad g(x)=x^3-4x$

$g(x)=0 \quad\therefore\quad x^3-4x=0 \quad\therefore\quad x=0,2,-2$

Hence, there are three real roots.

27. ***(2)*** $f(t)=g(t) \quad\therefore\quad f(t)=f(-t)$

$\therefore\qquad t=0 \qquad [\because f(t)$ is one-one function$]$

At $t=0, \qquad x=y=1$

$\because\qquad xy=1, \dfrac{dy}{dx}=\dfrac{-1}{x^2}$ and $\dfrac{d^2y}{dx^2}=\dfrac{2}{x^3}$

At $x=1, \qquad \dfrac{d^2y}{dx^2}=\dfrac{2}{1}=2$

28. ***(1)*** Let $t=\cos2\theta$. Then, $e^x=\dfrac{\sqrt{1+\cos2\theta}-\sqrt{1-\cos2\theta}}{\sqrt{1+\cos2\theta}+\sqrt{1-\cos2\theta}}$

$=\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\dfrac{1-\tan\theta}{1+\tan\theta}=\tan\left(\dfrac{\pi}{4}-\theta\right)$

$\tan\dfrac{y}{2}=\sqrt{\dfrac{1-\cos2\theta}{1+\cos2\theta}}=\tan\theta$

At $t=\dfrac{1}{2}, \qquad \cos2\theta=\dfrac{1}{2}$ or $\theta=\dfrac{\pi}{6}$

Then, $\qquad x=\log\tan\dfrac{\pi}{12}, y=\dfrac{\pi}{3}$

On differentiating w.r.t. θ, $e^x\dfrac{dx}{d\theta}=-\sec^2\left(\dfrac{\pi}{4}-\theta\right)$

and $\qquad \dfrac{1}{2}\sec^2\dfrac{y}{2}\times\dfrac{dy}{d\theta}=\sec^2\theta$

$\therefore\qquad \dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{2\sec^2\theta\cos^2\dfrac{y}{2}}{-e^{-x}\sec^2\left(\dfrac{\pi}{4}-\theta\right)}$

At $t=1/2$, i.e. $\theta=\dfrac{\pi}{6}, \dfrac{dy}{dx}=\dfrac{2\sec^2\dfrac{\pi}{6}\cos^2\dfrac{\pi}{6}}{-e^{-\log\tan\frac{\pi}{12}}\sec^2\dfrac{\pi}{12}}$

$\therefore\qquad \dfrac{dy}{dx}=\dfrac{2}{-\cot\dfrac{\pi}{12}\sec^2\dfrac{\pi}{12}}=-2\tan\dfrac{\pi}{12}\cos^2\dfrac{\pi}{12}$

$=-\sin\dfrac{\pi}{6}=-\dfrac{1}{2}$

But $\qquad \dfrac{dy}{dx}=\dfrac{-a}{2}$

$\therefore\qquad a=1$

29. ***(0)*** Given,

$$(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}+\frac{\sqrt{3}}{2}\sec^{-1}(2x)+2^x\tan[\log(x+2)]=0 \quad\ldots(i)$$

For $x=-1$, we have

$$(\sin y)^{\sin\left(\frac{\pi}{2}\right)}+\frac{\sqrt{3}}{2}\sec^{-1}(-2)+2^{-1}\tan[\log(-1+2)]=0$$

$\therefore\qquad (\sin y)^{-1}+\dfrac{\sqrt{3}}{2}\left(\dfrac{2\pi}{3}\right)+\dfrac{1}{2}\tan0°=0$

$\therefore\qquad \dfrac{1}{\sin y}=-\dfrac{\pi}{\sqrt{3}} \quad\therefore\quad \sin y=-\dfrac{\sqrt{3}}{\pi},$

when $\qquad x=-1$ …(ii)

Now, let $u=(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}$

Taking log on both sides, we get $\log u=\sin\dfrac{\pi x}{2}\log\sin y$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\times\frac{du}{dx}=\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\log\sin y+\cot y\frac{dy}{dx}\sin\left(\frac{\pi x}{2}\right)$$

$\therefore\qquad \dfrac{du}{dx}=(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}$

$$\left[\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\log\sin y+\sin\left(\frac{\pi x}{2}\right)\cot y\frac{dy}{dx}\right] \quad\ldots(iii)$$

Now, differentiating Eq. (i), we get

$$(\sin y)^{\sin\left(\frac{\pi x}{2}\right)}\left[\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\log\sin y+\sin\left(\frac{\pi x}{2}\right)\cot y\frac{dy}{dx}\right]+\frac{\sqrt{3}}{2x\sqrt{4x^2-1}}+2^x(\log2)\tan[\log(x+2)]+\frac{2^x\sec^2[\log(x+2)]}{x+2}=0$$

At $x=-1$ and $\sin y=-\dfrac{\sqrt{3}}{\pi}$, we get

$$\left(-\frac{\sqrt{3}}{\pi}\right)^{-1}\left[0+(-1)\sqrt{\frac{\pi^2}{3}-1}\left(\frac{dy}{dx}\right)_{x=-1}\right]+\frac{\sqrt{3}}{-2\sqrt{3}}+0+2^{-1}=0$$

$\therefore\qquad \dfrac{\pi}{\sqrt{3}\sqrt{3}}\sqrt{\pi^2-3}\left(\dfrac{dy}{dx}\right)_{x=-1}-\dfrac{1}{2}+\dfrac{1}{2}=0 \quad\therefore\quad \dfrac{dy}{dx}=0$

30. ***(5)*** $z=(\cos x)^5$; $y=\sin x$

$$\frac{dz}{dx}=-5\cos^4x\times\sin x; \frac{dy}{dx}=\cos x$$

$\therefore\qquad \dfrac{dz}{dy}=-5\cos^3x\times\sin x$

Now, $\qquad \dfrac{d^2z}{dy^2}=\dfrac{d}{dx}\left(\dfrac{dz}{dy}\right)\times\dfrac{dx}{dy}=-5\dfrac{d}{dx}[\cos^3x\times\sin x]\dfrac{1}{\cos x}$

$=-5[\cos^4x-3\sin^2x\times\cos^2x]\dfrac{1}{\cos x}$

$=-5(\cos^3x-3\sin^2x\times\cos x)$

$=-5[\cos^3x-3\cos x(1-\cos^2x)]$

$=-5(4\cos^3x-3\cos x)=-5\cos3x$

$\therefore\qquad \left.\dfrac{d^2z}{dy^2}\right|_{x=\frac{2\pi}{9}}=-5\cos120°=\dfrac{5}{2}$

$\therefore\qquad 2\dfrac{d^2z}{dy^2}=5$

OLYMPIAD PRACTICE

1. Let α and β be positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}$$

Find the minimum possible value of β.

Sol. We have,

$$\frac{77}{17} < \frac{\beta}{\alpha} < \frac{197}{43}$$

i.e. $$4 + \frac{9}{17} < \frac{\beta}{\alpha} < 4 + \frac{25}{43}$$

Thus, $4 < \frac{\beta}{\alpha} < 5$, since α and β are positive integers, we may write $\beta = 4\alpha + x$, where $0 < x < \alpha$.

Now, we get $4 + \frac{9}{17} < 4 + \frac{x}{\alpha} < 4 + \frac{25}{43}$

$\therefore$ $$\frac{9}{17} < \frac{x}{\alpha} < \frac{25}{43},$$

i.e. $$\frac{43x}{25} < \alpha < \frac{17x}{9}$$

We find the smallest value of x for which α becomes a well defined integer. For $x = 1, 2, 3$ the bounds of α are respectively $\left(1\frac{18}{25}, 1\frac{8}{9}\right), \left(3\frac{11}{25}, 3\frac{7}{9}\right), \left(5\frac{4}{25}, 5\frac{2}{3}\right)$. None of these pairs contain an integer between them.

For $x = 4$, we have $\frac{43x}{25} = 6\frac{22}{25}$ and $\frac{17x}{9} = 7\frac{5}{9}$

Hence, in this case $\alpha = 7$ and $\beta = 4\alpha + x = 28 + 4 = 32$

This is also the least possible value, because, if $x \geq 5$, then $\alpha > \frac{43x}{25} \geq \frac{43}{5} > 8$ and so $\beta > 37$.

Hence, the minimum possible value of β is 32.

2. Find all non-zero real numbers x, y, z which satisfy the system of equations.

$(x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) = xyz,$

$(x^4 + x^2y^2 + y^4)(y^4 + y^2z^2 + z^4)(z^4 + z^2x^2 + x^4) = x^3y^3z^3.$

Sol. Since, $xyz \neq 0$, so we can divide the second relation by the first observe that

$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2)$, holds for any x, y. Thus, we get

$$(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) = x^2y^2z^2$$

However, for any real numbers x, y, we have

$$x^2 - xy + y^2 \geq |xy|$$

Since, $x^2y^2z^2 = |xy||yz||zx|$, we get

$$|xy||yz||zx| = (x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2)$$
$$\geq |xy||yz||zx|$$

This is possible only, if

$$x^2 - xy + y^2 = |xy|,\ y^2 - yz + z^2 = |yz|,$$
$$z^2 - zx + x^2 = |zx|$$

hold simultaneously. However, $|xy| = \pm xy$

If $$x^2 - xy + y^2 = -xy$$

then $$x^2 + y^2 = 0 \text{ giving } x = y = 0$$

Since, we are looking for non-zero x, y, z, so we conclude that $x^2 - xy + y^2 = xy$ which is same as $x = y$. Using the other two relations, we also get $y = z$ and $z = x$. The first equation now gives $27x^6 = x^3$.

This gives $x^3 = \frac{1}{27}$ (since $x \neq 0$) or $x = \frac{1}{3}$. Thus, we have $x = y = z = \frac{1}{3}$. These also satisfy the second relation, as may be verified.

3. If a, b, c are positive real numbers such that $abc = 1$, then prove that $a^{b+c} \cdot b^{c+a} \cdot c^{a+b} \leq 1$.

Sol. Note that the inequality is symmetric in a, b, c so that we may assume that $a \geq b \geq c$.

Since, $abc = 1$, it follows that $a \geq 1$ and $c \leq 1$.

Using $b = \frac{1}{ac}$, we get

$$a^{b+c}b^{c+a}c^{a+b} = \frac{a^{b+c}c^{a+b}}{a^{c+a}c^{c+a}} = \frac{c^{b-c}}{a^{a-b}} \leq 1,$$

because $c \leq 1$, $b \leq c$, $a \geq 1$ and $a \geq b$.

4. Find all functions $f : R \to R$ such that

$$f(x^2 + y f(z)) = x f(x) + z f(y)$$

for all x, y, z in R.

Sol. We have,

$$f(x^2 + yf(z)) = xf(x) + zf(y) \quad \ldots(i)$$

Taking $x = y = 0$ in Eq. (i), we get $zf(0) = f(0), \forall\, z \in R$.

Hence, we obtain $f(0) = 0$

Taking $y = 0$ in Eq. (i), we get

$$f(x^2) = xf(x) \quad \ldots(ii)$$

Similarly, $x = 0$ in Eq. (i), we get

$$f(y\, f(z)) = z\, f(y) \quad \ldots(iii)$$

On putting $y = 1$ in Eq. (iii), we get

$$f(f(z)) = zf(1),\ \forall\ z \in R \quad \ldots(iv)$$

Now, using Eqs. (ii) and (iv), we get

$$f(x\, f(x)) = f(f(x^2)) = x^2 f\,(1) \quad \ldots(v)$$

On putting $y = z = x$ in Eq. (iii), we get

$$f(x\, f(x)) = x\, f(x) \quad \ldots(vi)$$

Comparing Eqs. (v) and (vi), it follows that $x^2 f(1) = x\, f(x)$. If $x \neq 0$, then $f(x) = cx$, for some constant c. Since, $f\,(0) = 0$, we have $f(x) = cx$ for $x = 0$ as well on substituting this in Eq. (i), we see that

$$c\,(x^2 + cyz) = cx^2 + cyz$$

or $\quad c^2 yz = cyz,\ \forall\ y, z \in R$

This implies that $c^2 = c$. Hence, $c = 0$ or 1.

We obtain $f(x) = 0$ for all x or $f(x) = x$ for all x. It is easy to verify that these two are solutions of the given equation.

5. Suppose $< x_1, x_2, \ldots, x_n, \ldots >$ is a sequence of positive real numbers such that $x_1 \geq x_2 \geq x_3 \geq \ldots \geq x_n \ldots$ and for all $n\ \dfrac{x_1}{1} + \dfrac{x_4}{2} + \dfrac{x_9}{3} + \ldots + \dfrac{x_{n^2}}{n} \leq 1$

Show that for all k the following inequality is satisfied $\dfrac{x_1}{1} + \dfrac{x_2}{2} + \dfrac{x_3}{3} + \ldots + \dfrac{x_k}{k} \leq 3$.

Sol. Let k be a natural number and n be the unique integer such that $(n-1)^2 \leq k < n^2$.

Then, we see that

$$\sum_{r=1}^{k} \frac{x_r}{r} \leq \left(\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3}\right) + \left(\frac{x_4}{4} + \frac{x_5}{5} + \ldots + \frac{x_8}{8}\right) + \ldots + \left(\frac{x_{(n-1)^2}}{(n-1)^2} + \ldots + \frac{x_k}{k} + \ldots + \frac{x_{n^2-1}}{n^2-1}\right)$$

$$\leq \left(\frac{x_1}{1} + \frac{x_1}{1} + \frac{x_1}{1}\right) + \left(\frac{x_4}{4} + \frac{x_4}{4} + \ldots + \frac{x_4}{4}\right) + \ldots + \left\{\frac{x_{(n-1)^2}}{(n-1)^2} + \ldots + \frac{x_{(n-1)^2}}{(n-1)^2}\right\}$$

$$= \frac{3x_1}{1} + \frac{5x_2}{4} + \ldots + \frac{(2n-1)x_{(n-1)^2}}{(n-1)^2}$$

$$= \sum_{r=1}^{n-1} \frac{(2r+1)x_{r^2}}{r^2} \leq \sum_{r=1}^{n-1} \frac{3r}{r^2} x_{r^2}$$

$$= 3 \sum_{r=1}^{n-1} \frac{x_{r^2}}{r} \leq 3$$

where the last inequality follows from the given hypothesis.

6. In an acute ΔABC, points D, E, F are located on the sides BC, CA, AB respectively such that

$$\frac{CD}{CE} = \frac{CA}{CB}, \frac{AE}{AF} = \frac{AB}{AC}, \frac{BF}{BD} = \frac{BC}{BA}.$$

Prove that AD, BE, CF are the altitudes of ΔABC.

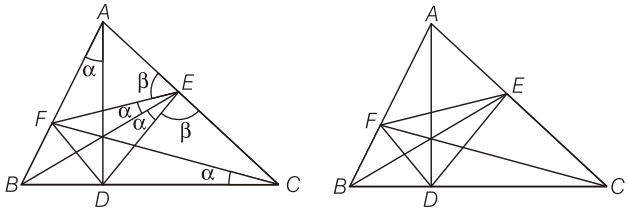
Sol. Put $CD = x$. Then, with usual notations, we get

$$CE = \frac{CD \cdot CB}{CA} = \frac{ax}{b}$$

Since, $\quad AE = AC - CE = b - CE$, we get

$$AE = \frac{b^2 - ax}{b}$$

$$AF = \frac{AE \cdot AC}{AB} = \frac{b^2 - ax}{c}$$



This in turn gives

$$BF = AB - AF = \frac{c^2 - b^2 + ax}{c}$$

Finally, we get

$$BD = \frac{c^2 - b^2 + ax}{a}$$

Using $\quad BD = a - x$, we get

$$x = \frac{a^2 - c^2 + b^2}{2a}$$

However, if L is the foot of perpendicular from A on to BC, then using Pythagoras theorem in ΔALB and ΔALC, we get

$$b^2 - LC^2 = c^2 - (a - LC)^2$$

which reduces to $LC = (a^2 - c^2 + b^2)/2a$, we conclude that $LC = DC$ proving $L = D$ or, we can also infer that $x = b\cos C$ from cosine rule in ΔABC. This implies that $CD = CL$, since $CL = b\cos C$ from right ΔALC. Thus, AD is altitude on to BC. Similar proof works for the remaining altitudes.

Alternately, we see that $CD \cdot CB = CE \cdot CA$, so that $ABDE$ is a cyclic quadrilateral. Similarly, we infer that $BCEF$ and $CAFD$ are also cyclic quadrilaterals.

Thus, $\quad \angle AEF = \angle B = \angle CED$

Moreover, $\angle BED = \angle DAF = \angle DCF$

$$= \angle BCF = \angle BEF$$

It follows that $\angle BEA = \angle BEC$ and hence each is a right angle thus proving that BE is an altitude Similarly, we prove that CF and AD are altitudes. (Note that the concurrence of the lines AD, BE, CF are not required.)

7. Suppose the integers 1, 2, 3, ...,10 are split into two disjoint collections a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 such that $a_1 < a_2 < a_3 < a_4 < a_5$; $b_1 > b_2 > b_3 > b_4 > b_5$

(i) Show that the larger number in any pair $\{a_j, b_j\}$, $1 \le j \le 5$, is atleast 6.

(ii) Show that $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$ for every such partition.

Sol. (i) Fix any pair $\{a_j, b_j\}$. We have $a_1 < a_2 < \ldots < a_{j-1} < a_j$ and $b_j > b_{j+1} > \ldots > b_5$. Thus, there are $j - 1$ numbers smaller than a_j and $5 - j$ numbers smaller than b_j. Together they account for $j - 1 + 5 - j = 4$ distinct numbers smaller than a_j as well as b_j. Hence, the larger of a_j and b_j is atleast 6.

(ii) The first part shows that the larger numbers in the pairs $\{a_j, b_j\}$, $1 \le j \le 5$ are 6, 7, 8, 9, 10 and the smaller numbers are 1, 2, 3, 4, 5. This implies that

$$|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 10 + 9 + 8 + 7 + 6 - (1 + 2 + 3 + 4 + 5) = 25$$

8. If n is an integer greater than 7, then prove that $\binom{n}{7} - \left[\frac{n}{7}\right]$ is divisible by 7. [Here, $\binom{n}{7}$ denotes the number of ways of choosing 7 objects from among n objects, also for any real number x, [x] denotes the greatest integer not exceeding x.]

Sol. We have,

$$\binom{n}{7} = \frac{n(n-1)(n-2)\ldots(n-6)}{7!}$$

In the numerator, there is a factor divisible by 7 and the other six factors leave the remainders 1, 2, 3, 4, 5, 6 in same order when divided by 7. Hence, the numerator may be written as $7k \cdot (7k_1 + 1) \cdot (7k_2 + 2)\ldots(7k_6 + 6)$.

Al so, we conclude that $\left[\frac{n}{p}\right] = k$, as in the set $\{n, n-1, \ldots, n-6\}$, $7k$ is the only number which is a multiple of 7. If the given number is called Q, then

$$Q = 7k \cdot \frac{(7k_1 + 1)(7k_2 + 2)\ldots(7k_6 + 6)}{7!} - k$$

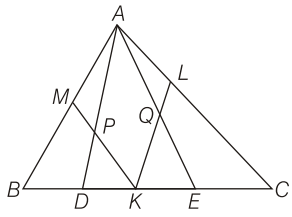
$$= k\left[\frac{(7k_1 + 1)\ldots(7k_6 + 6) - 6!}{6!}\right]$$

$$= \frac{k[7t + 6! - 6!]}{6!} = \frac{7tk}{6!}$$

We know that Q is an integer and so 6! divides $7tk$. Since, GCD (7 ! 6!) = 1, even after cancellation there is a factor of 7 still left in the numerator. Hence, 7 divides Q as desired.

9. The incircle of ΔABC touches the sides BC, CA and AB in K, L and M, respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of ΔABC.

Sol. Let AP, AQ produced meet BC in D, E, respectively.



Since, MK is parallel to AE, we have $\angle AEK = \angle MKB$. Since, $BK = BM$, both being tangents to the circle from B, $\angle MKB = \angle BMK$. This with the fact that MK is parallel to AE gives as $\angle AEK = \angle MAE$. This shows that $MAEK$ is an isosceles trapezoid. We conclude that $MA = KE$. Similarly, we can prove that $AL = DK$. But $AM = AL$. We get that $DK = KE$. Since, KP is parallel to AE, we get $DP = PA$ and similarly $EQ = QA$. This implies that PQ is parallel to DE and hence bisects AB, AC when produced.

10. Prove that there are infinitely many positive integers n such that $n(n+1)$ can be expressed as a sum of two positive squares in atleast two different ways. (Here, $a^2 + b^2$ and $b^2 + a^2$ are considered as same representation).

Sol. Let $Q = n(n + 1)$. It is convenient to choose $n = m^2$, for then Q is already a sum of two squares.

$$Q = m^2(m^2 + 1) = (m^2)^2 + m^2$$

If further m^2 itself is a sum of two squares say

$$m^2 = p^2 + q^2$$

then $Q = (p^2 + q^2)(m^2 + 1) = (pm + q)^2 + (p - qm)^2$

Note that the two representations for Q are distinct. Thus, for example, we may take $m = 5k$, $p = 3k$, $q = 4k$, where k varies over natural numbers. In this case $n = m^2 = 25k^2$ and $Q = (25k^2)^2 + (5k)^2 = (15k^2 + 4k)^2 + (20k^2 - 3k)^2$

As we vary k over natural numbers, we get infinitely many numbers of the form $n(n + 1)$ each of which can be expressed as a sum of two squares in two distinct ways.

Continued from Page 56

ANSWERS

1. 1 sq unit
2. $\frac{\pi}{6}$
3. 0.01
4. $n : 1$
5. $\frac{\pi}{2}$
7. decreasing
8. $f'(c) = 0$ and $f''(c) > 0$
9. 1
10. 2α
11. $\frac{-4}{125}$ rad/s
12. $x + y = 1$
14. $a = 2, b = -\frac{1}{2}$
15. 76 $y = \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4}$
16. decreasing on $(0, e) - \{1\}$
19. decreasing on $\left(\frac{-2}{3}(a^2 + b^2 + c^2), 0\right)$

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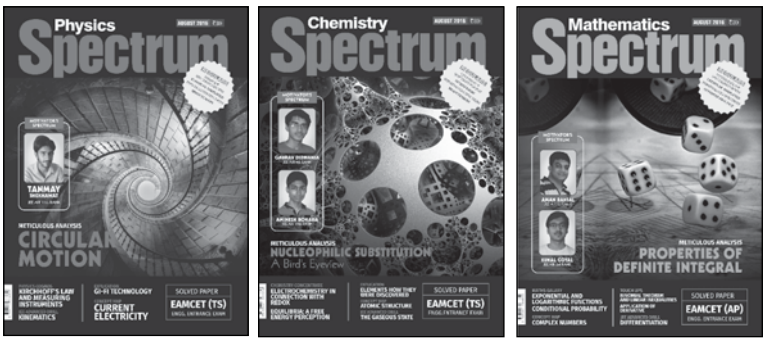

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MATHEMATICS @ SPECTRUM FOR YOU

In this section, we invite our readers to send their problems and we will try to provide the best possible solutions given by our expert panel.

Q.1 Sir, I am unable to find the solution of question given below. Please give me detailed solution of it.

"Let $\theta(x)$ denotes an angle measured in radians which is subtended by a fixed closed interval [1, 3] on the Y-axis at a point x on the positive X-axis. If x is the maximum value of $\theta(x)$. Then, find the maximum value of $\theta(x)$". **[Anil Kapoor, New Delhi]**

- Dear Anil, here I am giving the detailed solution of your question.

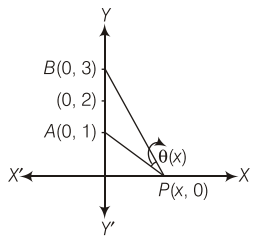
Since, it is given that closed interval [1, 3] on the Y-axis. So, let the points on Y-axis are $A(0, 1)$ and $B(0, 3)$ and let $P(x, 0)$ points be on X-axis.

So, $\theta(x) \angle APB = \angle XPA - \angle XPB$

$$\Rightarrow \quad \theta(x) = \tan^{-1}\left(\frac{1-0}{0-x}\right) - \tan^{-1}\left(\frac{3-0}{0-x}\right) \quad \left[\because \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

$$\Rightarrow \quad \theta(x) = \tan^{-1}\left(-\frac{1}{x}\right) - \tan^{-1}\left(-\frac{3}{x}\right)$$

$$\Rightarrow \quad \theta(x) = \tan^{-1}\left(\frac{3}{x}\right) - \tan^{-1}\left(-\frac{1}{x}\right) \quad [\because \tan^{-1}(-\theta) = -\tan^{-1}\theta]$$



On differentiating both sides w.r.t. x, we get

$$\frac{d\theta(x)}{dx} = \frac{1}{1+\left(\frac{3}{x}\right)^2} \times \frac{-3}{x^2} - \frac{1}{1+\frac{1}{x^2}} \times \frac{-1}{x^2}$$

$$\Rightarrow \quad \frac{d\theta(x)}{dx} = \frac{-3}{9+x^2} + \frac{1}{x^2+1}$$

$$= \frac{-3x^2 - 3 + 9 + x^2}{(x^2+1)(9+x^2)}$$

$$= \frac{-2x^2 + 6}{(x^2+1)(x^2+9)}$$

For maximum and minimum, take $\frac{d\theta(x)}{dx} = 0$

$$\therefore \quad \frac{-2x^2+6}{(x^2+1)(x^2+a)} = 0$$

$$\Rightarrow \quad -2x^2+6=0$$

$$\Rightarrow \quad x^2 = \frac{-6}{-2}$$

$$\Rightarrow \quad x = \pm\sqrt{3}$$

We will take $x = \sqrt{3}$ $\quad [\because x \neq -\sqrt{3}]$

It is given that $\theta(x)$ attains a maxima at x,

$$\therefore \quad \text{Max}\,\theta(x) = \tan^{-1}\frac{(2\sqrt{3})}{(\sqrt{3})^2+3}$$

$$= \tan^{-1}\frac{2\sqrt{3}}{6}$$

$$\Rightarrow \quad \text{Max}\,\theta(x) = \tan^{-1}\frac{\sqrt{3}}{3}$$

$$\Rightarrow \quad \text{Max}\,\theta(x) = \frac{\pi}{6}$$

02 Find the number of solutions of the equation $a^{f(x)} + g(x) = 0$, where $a > 0$, $g(x) \neq 0$ and $g(x)$ has minimum value $\frac{1}{4}$. Sir, I am unable to solve the above problem. Please give me detailed solution of it. **[Nitin, Kolkata]**

- Dear Nitin, here I am giving the detailed solution of your problem.

The given equation is

$$a^{f(x)} + g(x) = 0, \text{ where } a > 0,\ g(x) \neq 0$$

$$\Rightarrow \quad a^{f(x)} = -g(x)$$

$$\Rightarrow \quad a^{f(x)} \leq -\frac{1}{4} \qquad \left[\because \text{given, } g(x) \geq \frac{1}{4}\right]$$

But this is not possible as $a^{f(x)} > 0$ for all x.

Thus, the number of solutions is zero.

03 Hi, I am regular reader of your Mathematics Spectrum and please provide me the exact solution of my following problem.

"Prove that for any two real numbers a and b
$|\cos(a) - \cos(b)| \leq |a - b|$" **[Aryan, Delhi]**

- Dear Aryan, here I am giving the exact solution of your question, which would be the easiest to understand.

Let $f(x) = \cos x$, where $x \in [a, b]$

By using Lagrange's mean value theorem, we get

(i) $f(x)$ is continuous function on $[a, b]$

(ii) $f'(x) = -\sin x$ is exist on $]a, b[$

$$\therefore \quad f'(x) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad -\sin x = \frac{\cos(b)-\cos(a)}{b-a}$$

Taking mod sign on both sides, we get

$$|\sin x| = \left|\frac{\cos(b)-\cos(a)}{b-a}\right|$$

We know that, $|\sin x| \leq 1$

$$\therefore \quad \frac{|\cos(b)-\cos(a)|}{|b-a|} \leq 1$$

$$|\cos(a)-\cos(b)| \leq |a-b| \qquad \textbf{Hence proved.}$$

04 Suppose $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. If $|P(x)| \leq |e^{x-1} - 1|$ for all $x \geq 0$, prove that $|a_1 + 2a_2 + \ldots + na_n| \leq 1$

Sir, I am unable to solve the above problem. Please give me detailed solution of it. **[Amit, Mumbai]**

- Dear Amit, here I am giving the detailed solution of your problem.

We have,

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

On differentiating w.r.t. x both sides, we get

$$P'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}$$

$$\therefore \quad P'(1) = a_1 + 2a_2 + 3a_3 + \ldots + na_n$$

Now, we need to prove that $|P'(1)| \leq 1$

i.e. $\quad |a_1 + 2a_2 + \ldots + na_n| \leq 1$

It is given $\quad |P(x)| \leq |e^{x-1} - 1|$

$$\therefore \quad |P(1)| \leq |e^0 - 1|$$

$$\Rightarrow \quad |P(1)| \leq 0$$

$$\therefore \quad |P(1)| \leq 0 \Rightarrow |P(1)| = 0$$

Also, $-(e^{x-1} - 1) \leq P(x) \leq e^{x-1} - 1 \qquad [\because |x| < a \Rightarrow -a < x < a]$

On differentiating w.r.t. x, we get

$$-e^{x-1} \leq P'(x) \leq e^{x-1}$$

Put $\quad x = 1$, then we get

$$-1 \leq P'(1) \leq 1$$

$$\Rightarrow \quad |P'(1)| \leq 1$$

Hence, $|a_1 + 2a_2 + 3a_3 + \ldots + na_n| \leq 1$ **Hence proved.**

05 Sir, I am unable to find the solution of question given below. Please give me detailed solution of it.

'Let a, b and c are real numbers such that $0 < a < c < b < 1$, then prove that $a^2 + b^2 + ab = 3c^2$'. **[Navneet, Bihar]**

- Dear Navneet, here I am giving the detailed solution of your problem.

Let $f(x) = x^3$ be function such that $x \in [a, b]$.

(i) $f(x)$ is continuous function on $[a, b]$

(ii) $f(x)$ is differentiable function on $]a, b[$

By Lagrange's mean value theorem, then there exists a real number c which lying between $a < c < b$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}, \text{ where } c \in\ ]a, b[$$

$$\Rightarrow \quad 3c^2 = \frac{b^3-a^3}{b-a}$$

$$\Rightarrow \quad 3c^2 = \frac{(b-a)(b^2+a^2+ab)}{(b-a)}$$

Hence, $a^2 + b^2 + ab = 3c^2$ **Hence proved.**

Derivatives of Standard Functions

Differentiation of Some Important Functions

1. $\frac{d}{dx}(\text{constant}) = 0$
2. $\frac{d}{dx}(x^n) = nx^{n-1}$
3. $\frac{d}{dx}(\sin x) = \cos x$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\tan x) = \sec^2 x$
6. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$
7. $\frac{d}{dx}(\sec x) = \sec x \tan x$
8. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = a^x \log_e a,\ a > 0$
11. $\frac{d}{dx}(\log_e x) = \frac{1}{x},\ x > 0$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a},\ a > 0, a \neq 1$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$
14. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}},\ -1 < x < 1$
15. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
16. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}},\ |x| > 1$
18. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}},\ |x| > 1$
19. $\frac{d}{dx}(\sinh x) = \cosh x$
20. $\frac{d}{dx}(\cosh x) = \sinh x$
21. $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$
22. $\frac{d}{dx}(\coth x) = -\text{cosech}^2 x$
23. $\frac{d}{dx}(\text{sech } x) = -\text{sech } x \tanh x$
24. $\frac{d}{dx}(\text{cosech } x) = -\text{cosech } x \coth x$
25. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$
26. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
27. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{x^2-1}$
28. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$
29. $\frac{d}{dx}(\text{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$
30. $\frac{d}{dx}(\text{cosech}^{-1} x) = \frac{-1}{x\sqrt{1+x^2}}$

Fundamental Rules for Differentiation

1. $\frac{d}{dx}\{cf(x)\} = c\frac{d}{dx}f(x)$, where c is a constant.
2. $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$ [sum and difference rule]
3. $\frac{d}{dx}\{f(x)\cdot g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$ [product rule]
4. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{\{g(x)\}^2}$ [quotient rule]

Different Types of Differentiable Function

1. **Differentiation of Composite Function** (Chain Rule)
   If f and g are differentiable functions in their domain, then fog is also differentiable and $(fog)'x = f'[g(x)]\,g'(x)$
2. **Differentiation using Substitution**

	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
(ii)	$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $a\cot\theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
(iv)	$\sqrt{a+x}$ and $\sqrt{a-x}$	$x = a\cos 2\theta$
(v)	$a\sin x + b\cos x$	$a = r\cos\alpha,\ b = r\sin\alpha$
(vi)	$\sqrt{x-\alpha}$ and $\sqrt{\beta - x}$	$x = \alpha\sin^2\theta + \beta\cos^2\theta$
(vii)	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$

3. **Differentiation of Implicit Function**
   If $f(x, y) = 0$, then differentiate it with respect to x and collect the terms containing $\frac{dy}{dx}$ at one side and find $\frac{dy}{dx}$.
4. **Differentiation of Parametric Functions**
   If $x = f(t)$, $y = g(t)$, where t is a parameter, then
   $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{d}{dt}[g(t)]}{\frac{d}{dt}[f(t)]} = \frac{g'(t)}{f'(t)}$$
5. **Logarithmic Differentiation**
   (i) If a function is the product and quotient of functions such as $y = \frac{f_1(x) f_2(y) f_3(x)\ldots}{g_1(x) g_2(x) g_3(x)\ldots}$, we first take logarithm and then differentiate.
   (ii) If a function is in the form of exponent of a function over another function such as $[f(x)]^{g(x)}$ we first take logarithm and then differentiate.
6. **Differentiation of a Function with Respect to Another Function**
   Let $y = f(x)$ and $z = g(x)$, then differentiation of y with respect to z is
   $$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{f'(x)}{g'(x)}$$
7. **Leibnitz Theorem**
   If u and v are functions of x such that their nth derivative exist, then
   $$D^n(u\cdot v) = {}^nC_0(D^n u)\,v + {}^nC_1(D^{n-1}u)(Dv) + {}^nC_2(D^{n-2}u)(D^2 v) + \ldots + {}^nC_r D^{n-r}\cdot D^r v + \ldots + {}^nC_n(D^n v)$$
8. **Derivatives of Special Types of Function**
   (i) If $y = f(x)^{\{f(x)\}^{\cdots\infty}}$, then $\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)\{1 - y\log f(x)\}}$
   (ii) If $e^{g(y)} - e^{-g(y)} = 2f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{g'(y)}\cdot\frac{1}{\sqrt{1+\{f(x)\}^2}}$
   (iii) If $y = \sqrt{\frac{1-g(x)}{1+g(x)}}$, then $\frac{dy}{dx} = \frac{g'(x)}{[1-g(x)]^2}\cdot\sqrt{\frac{1-g(x)}{1+g(x)}}$
   (iv) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots\infty}}}$, then $\frac{dy}{dx} = \frac{f'(x)}{2y-1}$
   (v) If $\{f(x)\}^{g(y)} = e^{f(x) - g(y)}$, then $\frac{dy}{dx} = \frac{f'(x)\log f(x)}{g'(y)[1+\log f(x)]^2}$
   (vi) If $\{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}$, then
   $$\frac{dy}{dx} = \frac{g(y)}{f(x)}\cdot\frac{f'(x)}{g'(y)}\left[\frac{f(x)\log g(y) - g(y)}{g(y)\log f(x) - f(x)}\right]$$
9. **Differentiation of Integrable Function**
   If $g_1(x)$ and $g_2(x)$ are defined in $[a, b]$ differentiable at $x \in [a, b]$ and $f(t)$ is continuous for $g_1(a) \le f(t) \le g_2(b)$, then
   $$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(t)\,dt = f[g_2(x)]\frac{d}{dx}[g_2(x)] - f[g_1(x)]\frac{d}{dx}[g_1(x)]$$

EXEMPLAR S!MPL!FiED

EXTRACT OF EXTRA ORDINARY EXEMPLAR PROBLEM

RELATION, FUNCTION AND INVERSE TRIGONOMETRIC FUNCTIONS

Objective Type Questions

1. The maximum number of equivalence relations on the set $A=\{1,2,3\}$ are [Ex 1.3 Q. 30 Page 14]

(a) 1 (b) 2 (c) 3 (d) 5

2. If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is [Ex 1.3 Q. 35 Page 14]

(a) 720 (b) 120
(c) 0 (d) None of these

3. If $A=\{1,2,3,\ldots,n\}$ and $B=\{a,b\}$. Then, the number of surjections from A into B is [Ex 1.3 Q. 36 Page 14]

(a) ${}^{n}P_{2}$ (b) $2^{n}-2$
(c) $2^{n}-1$ (d) None of these

4. If $f:[2,\infty)\to R$ be the function defined by $f(x)=x^{2}-4x+5$, then the range of f is [Ex 1.3 Q. 44 Page 16]

(a) R (b) $[1,\infty)$ (c) $[4,\infty)$ (d) $[5,\infty)$

5. If $f:R\to R$ be defined by

$$f(x)=\begin{cases}2x, & x>3\\ x^{2}, & 1<x\le 3\\ 3x, & x\le 1\end{cases}$$

Then, $f(-1)+f(2)+f(4)$ is [Ex 1.3 Q. 46 Page 16]

(a) 9 (b) 14
(c) 5 (d) None of these

6. If $3\tan^{-1}x+\cot^{-1}x=\pi$, then x is equal to [Ex 2.3 Q. 22 Page 37]

(a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

7. The domain of the function defined by $f(x)=\sin^{-1}\sqrt{x-1}$ is [Ex 2.3 Q. 25 Page 38]

(a) [1, 2] (b) [−1, 1]
(c) [0, 1] (d) None of these

8. The value of $\sin[2\tan^{-1}(0.75)]$ is [Ex 2.3 Q. 27 Page 38]

(a) 0.75 (b) 1.5 (c) 0.96 (d) sin1.5

9. If $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$, then $\cot^{-1}x+\cot^{-1}y$ is equal to [Ex 2.3 Q. 30 Page 38]

(a) $\frac{\pi}{5}$ (b) $\frac{2\pi}{5}$
(c) $\frac{3\pi}{5}$ (d) π

10. If $\cos^{-1}\alpha+\cos^{-1}\beta+\cos^{-1}\gamma=3\pi$, then $\alpha(\beta+\gamma)+\beta(\gamma+\alpha)+\gamma(\alpha+\beta)$ is equal to [Ex 2.3 Q. 35 Page 39]

(a) 0 (b) 1 (c) 6 (d) 12

11. The number of real solutions of the equation $\sqrt{1+\cos 2x}=\sqrt{2}\cos^{-1}(\cos x)$ in $\left[\frac{\pi}{2},\pi\right]$ is [Ex 2.3 Q. 36 Page 39]

(a) 0 (b) 1 (c) 2 (d) ∞

12. If $\cos^{-1}x>\sin^{-1}x$, then [Ex 2.3 Q. 37 Page 39]

(a) $\frac{1}{\sqrt{2}}<x\le 1$ (b) $0\le x<\frac{1}{\sqrt{2}}$
(c) $-1\le x<\frac{1}{\sqrt{2}}$ (d) $x>0$

Subjective Type Questions

13. Let $A=\{a,b,c\}$ and the relation R be defined on A as follows $R=\{(a,a),(b,c),(a,b)\}$.

Then, write minimum number of ordered pairs to be added in R to move R reflexive and transitive.
[Ex 1.3 Q. 1 Page 11]

14. Let the function $f:R\to R$ be defined by $f(x)=\cos x$, $\forall\, x\in R$. Show that f is neither one-one nor onto.
[Ex 1.3 Q. 11 Page 11]

15. Let $f:R\to R$ be the function, defined by $f(x)=\dfrac{1}{2-\cos x}$, $\forall\, x\in R$. Then, find the range of f.
[Ex 1.3 Q. 14 Page 12]

16. Find the real solution of

$$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$$

[Ex 2.3 Q. 7 Page 36]

17. Prove that $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x^2$
[Ex 2.3 Q. 12 Page 36]

18. Show that $\tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right)=\dfrac{4-\sqrt{7}}{3}$ and justify why the other value $\dfrac{4+\sqrt{7}}{3}$ is ignored? [Ex 2.3 Q. 18 Page 37]

19. If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with common difference d, then evaluate the following expression.

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$

[Ex 2.3 Q. 19 Page 37]

20. Prove that $\cot\left(\dfrac{\pi}{4}-2\cot^{-1}3\right)=7$ [Ex 2.3 Q. 3 Page 35]

Answers with Explanation

1. ***(d)*** Given that, $A=\{1, 2, 3\}$

Now, number of equivalence relations on the set $A=\{1,2,3\}$ are as follows

$R_1=\{(1,1),(2,2),(3,3)\}$
$R_2=\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$
$R_3=\{(1,1),(2,2),(3,3),(1,3),(3,1)\}$
$R_4=\{(1,1),(2,2),(3,3),(2,3),(3,2)\}$
$R_5=\{(1,2,3)\Leftrightarrow A\times A=A^2\}$

$\therefore$ Maximum number of equivalence relation on the set $A=\{1,2,3\}=5$

2. ***(c)*** We know that, if A and B are two non-empty finite set containing m and n elements respectively, then the number of one-one and onto mapping from A to B is

$$\begin{cases} n!, & \text{if } m=n \\ 0, & \text{if } m\neq n \end{cases}$$

Given that, $m=5$ and $n=6$

$\therefore\quad m\neq n$

Number of mappings $=0$

3. ***(d)*** Given that, $A=\{1, 2, 3,\ldots,n\}$ and $B=\{a,b\}$

We know that, it A and B are two non-empty finite sets containing m and n elements respectively, then the number of surjection from A into B is

$$\begin{cases} {}^nC_m\times m!, & \text{if } n\geq m \\ 0, & \text{if } n<m \end{cases}$$

Hence, $m=2$

$\therefore$ Number of surjection from A into B is

$$={}^nC_2\times 2!=\frac{n!}{2!(n-2)!}\times 2!$$

$$=\frac{n(n-1)(n-2)!}{(n-2)!}=n^2-n$$

4. ***(b)*** Given that,

$$f(x)=x^2-4x+5$$

Let $\quad y=x^2-4x+5$

$\Rightarrow\quad y=x^2-4x+4+1$

$\Rightarrow\quad y=(x-2)^2+1\Rightarrow(x-2)^2=y-1$

$\Rightarrow\quad x-2=\sqrt{y-1}\quad\Rightarrow\quad x=2+\sqrt{y-1}$

$\therefore\quad y-1\geq 0\quad\Rightarrow\quad y\geq 1$

Range $=[1,\infty)$

5. ***(a)*** Given that,

$$f(x)=\begin{cases} 2x, & x>3 \\ x^2, & 1<x\leq 3 \\ 3x, & x\leq 1 \end{cases}$$

$\Rightarrow\quad f(-1)=3(-1)=-3$

$f(2)=(2)^2=4$

$f(4)=2\times 4=8$

$\therefore\quad f(-1)+f(2)+f(4)=-3+4+8=9$

6. ***(b)*** Given that,

$$3\tan^{-1}x+\cot^{-1}x=\pi$$

$\Rightarrow\quad 2\tan^{-1}x+\tan^{-1}x+\cot^{-1}x=\pi$

$\Rightarrow\quad 2\tan^{-1}x+\dfrac{\pi}{2}=\pi\quad\left[\because\tan^{-1}x+\cot^{-1}x=\dfrac{\pi}{2}\right]$

$\Rightarrow\quad 2\tan^{-1}x=\dfrac{\pi}{2}\Rightarrow\tan^{-1}x=\dfrac{\pi}{4}$

$\Rightarrow\quad x=\tan\dfrac{\pi}{4}$

$\Rightarrow\quad x=1$

7. ***(a)*** $\because\ f(x)=\sin^{-1}\sqrt{x-1}$

$\Rightarrow\quad \sqrt{x-1}\geq 0$ and $-1\leq\sqrt{x-1}\leq 1$

$\Rightarrow\quad 0\leq x-1\leq 1\Rightarrow 1\leq x\leq 2$

$\therefore\quad x\in[1,2]$

8. *(c)* We have, $\sin[2\tan^{-1}(0.75)]=\sin\left(2\tan^{-1}\frac{3}{4}\right)$

$$=\sin\left(\sin^{-1}\frac{2\times\frac{3}{4}}{1+\frac{9}{16}}\right)=\sin\left(\sin^{-1}\frac{\frac{3}{2}}{\frac{25}{16}}\right)$$

$$=\sin\left[\sin^{-1}\left(\frac{48}{50}\right)\right]=\sin\left[\sin^{-1}\left(\frac{24}{25}\right)\right]$$

$$=\frac{24}{25}=0.96$$

9. *(a)* We have, $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$

$\Rightarrow \frac{\pi}{2}-\cot^{-1}x+\frac{\pi}{2}-\cot^{-1}y=\frac{4\pi}{5}$

$\Rightarrow -(\cot^{-1}x+\cot^{-1}y)=\frac{4\pi}{5}-\pi$

$\Rightarrow \cot^{-1}x+\cot^{-1}y=-\left(-\frac{\pi}{5}\right)$

$\Rightarrow \cot^{-1}x+\cot^{-1}y=\frac{\pi}{5}$

10. *(c)* We have, $\cos^{-1}\alpha+\cos^{-1}\beta+\cos^{-1}\gamma=3\pi$

We know that, $0\le\cos^{-1}x\le\pi$

$\Rightarrow \cos^{-1}\alpha+\cos^{-1}\beta+\cos^{-1}\gamma=3\pi$

If and only if $\cos^{-1}\alpha=\cos^{-1}\beta=\cos^{-1}\gamma=\pi$

$\Rightarrow \cos\pi=\alpha=\beta=\gamma \Rightarrow -1=\alpha=\beta=\gamma$

$\therefore \alpha(\beta+\gamma)+\beta(\gamma+\alpha)+\gamma(\alpha+\beta)$

$=-1(-1-1)-1(-1-1)-1(-1-1)$

$=2+2+2=6$

11. *(a)* We have, $\sqrt{1+\cos 2x}=\sqrt{2}\cos^{-1}(\cos x), \left[\frac{\pi}{2},\pi\right]$

$\Rightarrow \sqrt{2\cos^2 x}=\sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow \sqrt{2}\cos x=\sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow \cos x=\cos^{-1}(\cos x)$

$\Rightarrow \cos x=x \quad [\because \cos^{-1}(\cos x)=x]$

which is not true for any real value of x.

Hence, there is no solution possible for the given equation.

12. *(c)* We have, $\cos^{-1}x>\sin^{-1}x$, where $x\in[-1,1]$

$\Rightarrow x<\cos(\sin^{-1}x)$

$\Rightarrow x<\cos\left(\cos^{-1}\sqrt{1-x^2}\right) \quad [\because \sin^{-1}x=\cos^{-1}\sqrt{1-x^2}]$

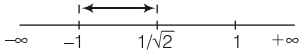
$\Rightarrow x<\sqrt{1-x^2} \Rightarrow x^2<1-x^2$

$\Rightarrow 2x^2<1 \Rightarrow x^2<\frac{1}{2}$

$\Rightarrow x<\pm\left(\frac{1}{\sqrt{2}}\right)$

Also, $-1\le x\le 1$

$\therefore -1\le x<\frac{1}{\sqrt{2}}$



13. Given relation, $R=\{(a,a),(b,c),(a,b)\}$

To make R is reflexive we must add (b,b) and (c,c) to R. Also, to make R is transitive we must add (a,c) to R. So, minimum number of ordered pair is to be added are (b,b), (c,c), (a,c).

14. Given function, $f(x)=\cos x, \forall\, x\in R$

Now, $f\left(\frac{\pi}{2}\right)=\cos\frac{\pi}{2}=0$

Also, $f\left(-\frac{\pi}{2}\right)=\cos\left(-\frac{\pi}{2}\right)=0 \Rightarrow f\left(\frac{\pi}{2}\right)=f\left(-\frac{\pi}{2}\right)$

But $\frac{\pi}{2}\neq-\frac{\pi}{2}$

So, $f(x)$ is not one-one.

Now, $f(x)=\cos x, \forall\, x\in R$ is not onto as there is no pre-image for any real number which does not belongs to the interval $[-1,1]$, the range of $\cos x$.

15. Given function, $f(x)=\frac{1}{2-\cos x}, \forall\, x\in R$

Let $y=\frac{1}{2-\cos x} \Rightarrow 2y-y\cos x=1$

$\Rightarrow y\cos x=2y-1 \Rightarrow \cos x=\frac{2y-1}{y}=2-\frac{1}{y}$

Now, $-1\le\cos x\le 1$

$\Rightarrow -1\le 2-\frac{1}{y}\le 1 \Rightarrow -3\le-\frac{1}{y}\le-1$

$\Rightarrow 1\le\frac{1}{y}\le 3 \Rightarrow \frac{1}{3}\le y\le 1$

So, range of $f(x)$ is $\left[\frac{1}{3},1\right]$.

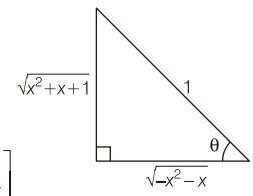
16. We have, $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$ ...(i)

Let $\sin^{-1}(\sqrt{x^2+x+1})=\theta$

$\Rightarrow \sin\theta=\frac{\sqrt{x^2+x+1}}{1}$

$\Rightarrow \tan\theta=\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}}$

$\Rightarrow \theta=\tan^{-1}\left[\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}}\right]$



On putting the value of θ in Eq. (i), we get

$$\tan^{-1}\sqrt{x(x+1)}+\tan^{-1}\left[\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}}\right]=\frac{\pi}{2}$$

We know that, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1$

$$\therefore \tan^{-1}\left[\frac{\sqrt{x(x+1)}+\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}}}{1-\sqrt{x(x+1)}\cdot\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}}}\right]=\frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\sqrt{x^2+x}+\frac{\sqrt{x^2+x+1}}{\sqrt{-(x^2+x)}}}{1-\sqrt{(x^2+x)}\cdot\frac{(x^2+x+1)}{-1(x^2+x)}}\right]=\frac{\pi}{2}$$

$$\Rightarrow \frac{x^2+x+\sqrt{-(x^2+x+1)}}{1-\sqrt{-(x^2+x+1)}\sqrt{x^2+x}}=\tan\frac{\pi}{2}=\frac{1}{0}$$

$\Rightarrow \quad 1-\sqrt{-(x^2+x+1)}\sqrt{x^2+x}=0$

$\Rightarrow \quad -(x^2+x+1)=1 \text{ or } x^2+x=0$

$\Rightarrow \quad -x^2-x-1=1 \text{ or } x(x+1)=0$

$\Rightarrow \quad x^2+x+2=0 \text{ or } x(x+1)=0$

$\therefore \quad x=\dfrac{-1\pm\sqrt{1-4\times2}}{2}$ or $x=0,\ x=-1$

For real solution, we have $x=0,-1$

17. We have, $\text{LHS}=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Put $x^2=\cos2\theta \Rightarrow \theta=\dfrac{1}{2}\cos^{-1}x^2$

$\therefore \quad \text{LHS}=\tan^{-1}\left(\dfrac{\sqrt{1+\cos2\theta}+\sqrt{1-\cos2\theta}}{\sqrt{1+\cos2\theta}-\sqrt{1-\cos2\theta}}\right)$

$=\tan^{-1}\left(\dfrac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$

$=\tan^{-1}\left(\dfrac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$

$=\tan^{-1}\left(\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)=\tan^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$

$=\tan^{-1}\left(\dfrac{\tan\frac{\pi}{4}+\tan\theta}{\tan\frac{\pi}{4}-\tan\theta}\right)=\tan^{-1}\left[\tan\left(\dfrac{\pi}{4}+\theta\right)\right]$

$=\dfrac{\pi}{4}+\theta=\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x^2$

$=\text{RHS}$ **Hence proved.**

18. We have, $\tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right)=\dfrac{4-\sqrt{7}}{3}$

$\therefore \quad \text{LHS}=\tan\left[\dfrac{1}{2}\sin^{-1}\left(\dfrac{3}{4}\right)\right]$

Let $\dfrac{1}{2}\sin^{-1}\left(\dfrac{3}{4}\right)=\theta \Rightarrow \sin^{-1}\left(\dfrac{3}{4}\right)=2\theta$

$\Rightarrow \quad \sin2\theta=\dfrac{3}{4} \Rightarrow \dfrac{2\tan\theta}{1+\tan^2\theta}=\dfrac{3}{4}$

$\Rightarrow \quad 3+3\tan^2\theta=8\tan\theta$

$\Rightarrow \quad 3\tan^2\theta-8\tan\theta+3=0$

$\Rightarrow \quad \tan\theta=\dfrac{8\pm\sqrt{(-8)^2-4(3)(3)}}{2\times3}$

$\Rightarrow \quad \tan\theta=\dfrac{8\pm\sqrt{64-36}}{6}=\dfrac{8\pm\sqrt{28}}{6}$

$\Rightarrow \quad \tan\theta=\dfrac{2\,[4\pm\sqrt{7}]}{6}$

$\Rightarrow \quad \tan\theta=\dfrac{4\pm\sqrt{7}}{3}$

$\Rightarrow \quad \tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right)=\dfrac{4\pm\sqrt{7}}{3}$

Since, $\dfrac{4+\sqrt{7}}{3}>1$, but $\tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right)\in[-1,1]$

$\therefore \quad \tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right)=\dfrac{4-\sqrt{7}}{3}$ **Hence proved.**

19. We have, $a_1=a,\ a_2=a+d,\ a_3=a+2d,\ldots$

and $d=a_2-a_1=a_3-a_2=a_4-a_3=\ldots=a_n-a_{n-1}$

Given that,

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]$$

$$=\tan\left[\tan^{-1}\left(\frac{a_2-a_1}{1+a_2a_1}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right)+\ldots+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right)\right]$$

$$=\tan[(\tan^{-1}a_2-\tan^{-1}a_1)+(\tan^{-1}a_3-\tan^{-1}a_2)+\ldots+(\tan^{-1}a_n-\tan^{-1}a_{n-1})]$$

$$=\tan(\tan^{-1}a_n-\tan^{-1}a_1)$$

$$=\tan\left[\tan^{-1}\left(\frac{a_n-a_1}{1+a_1a_n}\right)\right]$$

$$=\frac{a_n-a_1}{1+a_1a_n}$$

20. We have to prove,

$\cot\left(\dfrac{\pi}{4}-2\cot^{-1}3\right)=7$

i.e. to prove $\dfrac{\pi}{4}-2\cot^{-1}3=\cot^{-1}7$

or $2\cot^{-1}3=\dfrac{\pi}{4}-\cot^{-1}7$

or $2\tan^{-1}\left(\dfrac{1}{3}\right)=\dfrac{\pi}{4}-\tan^{-1}\left(\dfrac{1}{7}\right)$

or $2\tan^{-1}\left(\dfrac{1}{3}\right)+\tan^{-1}\left(\dfrac{1}{7}\right)=\dfrac{\pi}{4}$

Consider $\text{LHS}=\tan^{-1}\left(\dfrac{2\times1/3}{1-(1/3)^2}\right)+\tan^{-1}\left(\dfrac{1}{7}\right)$

$=\tan^{-1}\left(\dfrac{2/3}{8/9}\right)+\tan^{-1}\left(\dfrac{1}{7}\right)$

$=\tan^{-1}\left(\dfrac{3}{4}\right)+\tan^{-1}\left(\dfrac{1}{7}\right)$

$=\tan^{-1}\left(\dfrac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right)$

$=\tan^{-1}\left(\dfrac{\left(\frac{21+4}{28}\right)}{\left(\frac{28-3}{28}\right)}\right)$

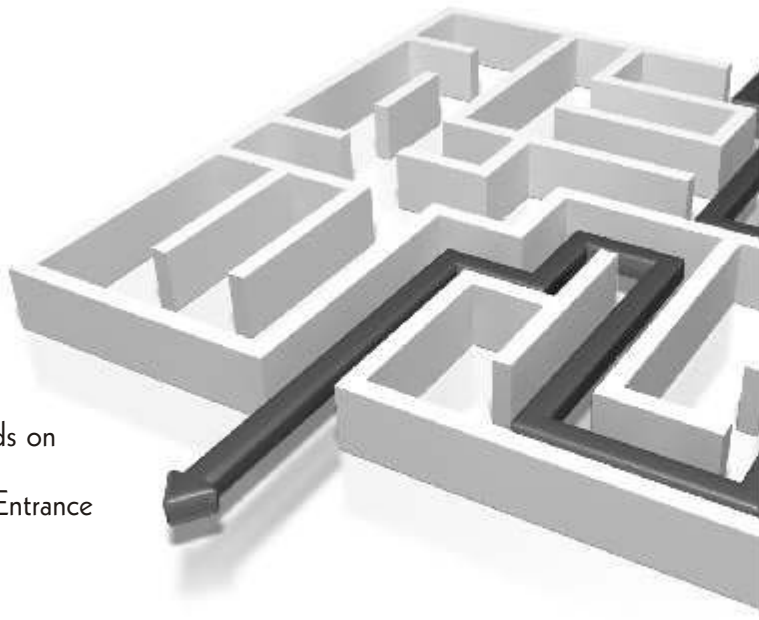
$=\tan^{-1}\left(\dfrac{25}{25}\right)$

$=\tan^{-1}1$

$=\dfrac{\pi}{4}$

$=\text{RHS}$ **Hence proved.**

SHORTCUT METHODS



Here, we are introducing some important shortcut methods on Circle and Parabola to save your precious time in tedious calculations based problems asked in various Engineering Entrance exams. Use these shortcuts, to practice questions and be acquainted with it.

1 The equation of the circumcircle of the triangle formed by the line $ax + by + c = 0$ with the coordinate axes is

$$ab(x^2 + y^2) + c(bx + ay) = 0$$

Example *The equation of the circumcircle of the triangle formed by line* $2x + 3y - 4 = 0$ *with coordinate axes is*

(a) $6(x^2 + y^2) - 12x - 8y = 0$ *(b)* $6(x^2 + y^2) + 12x + 8y = 0$
(c) $x^2 + y^2 + 2x + 3y = 0$ *(d)* None of these

Sol. ***(a)*** Equation of circumcircle of the triangle

$$ab(x^2 + y^2) + c(bx + ay) = 0$$
$$(2)(3)(x^2 + y^2) + (-4)(3x + 2y) = 0$$
$$6(x^2 + y^2) - 12x - 8y = 0$$

2 Tangents are drawn from the point $P(h, k)$ to the circle $x^2 + y^2 = a^2$. The area of the triangle formed by them and their chord of contact is $\frac{a(h^2 + k^2 - a^2)^{1/2}}{h^2 + k^2}$.

Example *From the point* $P(3, 4)$ *tangents* PA *and* PB *are drawn to the circle* $x^2 + y^2 = 3^2$. *The area of* ΔPAB *in sq units is*

(a) $\frac{16}{5}$ sq units *(b)* $\frac{5}{4}$ sq units
(c) $\frac{12}{25}$ sq unit *(d)* None of these

Sol. ***(c)*** Point $P(3, 4)$ and equation of circle is $x^2 + y^2 = 3^2$

$$\text{Area of } \Delta PAB = \frac{a(h^2 + k^2 - a^2)^{\frac{1}{2}}}{h^2 + k^2} = \frac{3(3^2 + 4^2 - 3^2)^{\frac{1}{2}}}{3^2 + 4^2}$$
$$= \frac{3 \times 4}{25} = \frac{12}{25} \text{ sq unit}$$

3 If two circles of radii r_1 and r_2 intersect orthogonally, then length of the common chord $= \frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$

Example *The length of the common chord of two orthogonal circles of radii* 15 *and* 20, *whose centres are* 25 *units apart, is*

(a) 24 *(b)* 25
(c) 15 *(d)* 20

Sol. ***(a)*** Length of the common chord

$$= \frac{2rR}{\sqrt{r^2 + R^2}}$$
$$= \frac{2 \times 15 \times 20}{25} = 24$$

4 If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circle $x^2 + y^2 + 2g'x + 2f'y + c' = 0$, then the length of the common chord of these two circles is $2\sqrt{g'^2 + f'^2 - c'}$.

Example *If the circle* $x^2 + y^2 + 4x + 22y + 8 = 0$ *bisects the circumference of the circle* $x^2 + y^2 - 2x + 8y + 1 = 0$, *then common chord of these two circles*

(a) 10 *(b)* 8
(c) 9 *(d)* 5

Sol. ***(b)*** Common chord $= 2\sqrt{g'^2 + f'^2 - c'}$

Here, $g'^2 = 1, f'^2 = 16, c' = 1$

$\therefore$ Common chord $= 2\sqrt{1 + 16 - 1}$
$= 2\sqrt{16} = 8$

5 The area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$, where y_1, y_2, y_3 are the ordinates of the angular points.

Example *Area of the triangle inscribed in the parabola $y^2 = 4ax$, such that one vertex of this triangle coincide with the vertex of the parabola and two other vertices are extremities of the latusrectum is*

(a) $3a^2$ sq units (b) $2a^2$ sq units
(c) a^2 sq units (d) $4a$ sq units

Sol. **(b)** Area of the triangle whose ordinates are 0, $2a$ and $-2a$ is

$$= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$$

$$= \frac{1}{8a}(0 - 2a)(2a + 2a)(-2a - 0) = \frac{16a^3}{8a} = 2a^2 \text{ sq units}$$

6 The equation of the parabola whose vertex and focus are on X-axis at a distance a and a' respectively from the origin is $y^2 = 4(a' - a)(x - a)$.

Example *Equation of the parabola whose vertex and focus are at a distance 2 and 4 units away from origin on X-axis is*

(a) $y^2 = 4x$ (b) $y^2 = 4(x - 2)$
(c) $y^2 = 8(x - 2)$ (d) $y^2 = 8x$

Sol. **(c)** $y^2 = 4(a' - a)(x - a)$

Here, $a = 2$ and $a' = 4$

$\therefore$ $y^2 = 4(4 - 2)(x - 2) \Rightarrow y^2 = 8(x - 2)$

7 If the straight line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$, then $l \times n = am^2$.

Example *If $2x + 3y + 5 = 0$ touches the parabola $y^2 = 4ax$, then a is*

(a) 10 (b) $\frac{10}{9}$ (c) 9 (d) $\frac{9}{10}$

Sol. **(b)** Here, $l = 2$, $m = 3$ and $n = 5$

$\therefore$ $l \times n = am^2$ [when line touches the parabola]

$\therefore$ $2(5) = a(9) \Rightarrow a = \frac{10}{9}$

8 Point of intersection of tangents at two points $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $R\{at_1t_2, a(t_1 + t_2)\}$.

Example *If the tangents to the parabola $y^2 = 4ax$ at (x_1, y_1) and (x_2, y_2) intersect at (x_3, y_3). Then,*

(a) x_1, x_2, x_3 are in GP (b) x_1, x_2, x_3 are in AP
(c) y_1, y_2, y_3 are in GP (d) y_1, y_2, y_3 are in AP

Sol. **(a, d)** From above shortcut it is clear that intersection points of tangent R has x-coordinate in GP and y-coordinate in AP.

9 Angle θ between tangent at two points $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is given by $\tan\theta = \left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$.

Example *Angle between the tangent drawn at $(2, 4)$ and $(1, 2\sqrt{2})$ to the parabola $y^2 = 8x$ is*

(a) $\tan^{-1}\left|\frac{1 - \sqrt{2}}{1 + \sqrt{2}}\right|$ (b) $\tan^{-1}\left|\frac{1 + \sqrt{2}}{1 - \sqrt{2}}\right|$
(c) $\tan^{-1}\sqrt{2}$ (d) None of these

Sol. **(a)** Here, $a = 2$

$$4t_1 = 4 \Rightarrow t_1 = 1$$

Also, $$4t_2 = 2\sqrt{2} \Rightarrow t_2 = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \tan\theta = \left|\frac{\frac{1}{\sqrt{2}} - 1}{1 + \frac{1}{\sqrt{2}} \cdot 1}\right| = \left|\frac{1 - \sqrt{2}}{1 + \sqrt{2}}\right| \Rightarrow \theta = \tan^{-1}\left|\frac{1 - \sqrt{2}}{1 + \sqrt{2}}\right|$$

10 The angle of intersection of two parabola $y^2 = 4ax$ and $x^2 = 4by$ is given by $\tan^{-1}\left|\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}} + b^{\frac{2}{3}})}\right|$.

Example *Angle between the parabola $x^2 = 4y$ and $y^2 = 4x$ is*

(a) $\tan^{-1}\left|\frac{4}{3}\right|$ (b) $\tan^{-1}\left|\frac{3}{4}\right|$ (c) $\tan^{-1}\left|\frac{1}{2}\right|$ (d) $\tan^{-1}|2|$

Sol. **(b)** Here, $a = b = 1$

$$\theta = \tan^{-1}\left|\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}} + b^{\frac{2}{3}})}\right| = \tan^{-1}\left|\frac{3}{2(1 + 1)}\right| = \tan^{-1}\left|\frac{3}{2 \times 2}\right| = \tan^{-1}\left|\frac{3}{4}\right|$$

11 If normal at point t_1 meets the parabola $y^2 = 4ax$ again at point t_2, then $t_2 = -t_1 - \frac{2}{t_1}$. Also, normal at t_1 and t_2 meet the parabola at same point, then $t_1t_2 = 2$.

Example *If the normal at $(ap^2, 2ap)$ on the parabola $y^2 = 4ax$ meet the parabola again at $(aq^2, 2aq)$, then*

(a) $p^2 + pq - 2 = 0$ (b) $p^2 - pq + 2 = 0$
(c) $p^2 + pq + 2 = 0$ (d) $p^2 + pq + 1 = 0$

Sol. **(c)** Here, $t_1 = p$ and $t_2 = q$

$\because$ $$t_2 = -t_1 - \frac{2}{t_1}$$

$\therefore$ $$q = -p - \frac{2}{p} \Rightarrow pq = -p^2 - 2$$

$\Rightarrow$ $p^2 + pq + 2 = 0$

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1. Which of the following is true?

(a) $\log\left(\frac{x+y}{2}\right) \le \frac{\log x + \log y}{2}$, $\forall\ x, y > 0$

(b) $e^{\frac{x+y}{2}} \le \frac{e^x + e^y}{2}$, $\forall\ x, y > 0$

(c) $\sin\left(\frac{x+y}{2}\right) \le \frac{\sin x + \sin y}{2}$, $\forall\ x, y > 0$

(d) $\frac{(x+y)^k}{2^k} \ge \max\{x^k, y^k\}$, $\forall\ x, y > 0$ and $k > 1$

2. Let $f(x) = \sin x - x + \frac{x^3}{3!}$ and $g(x) = \cos x - 1 + \frac{x^2}{2!}$ for $x \in R$. Which of the following statement is correct?

(a) $f(x) \le 0, \forall\ x < 0$

(b) g is an increasing function on $[0, \infty)$

(c) g is a decreasing function on $[0, \infty)$

(d) f is a decreasing function on $[0, \infty)$

3. Let $x_1, x_2, \ldots$ be $N(1,1)$ random variables. Let $S_n = x_1^2 + x_2^2 + \ldots + x_n^2$ for $n \ge 1$. Then, $\lim_{n\to\infty} \frac{\text{Var}(S_n)}{n}$ is

(a) 4 *(b)* 6

(c) 1 *(d)* 0

4. Let $I = \{1\} \cup \{2\} \cup \{3\} \subset R$ for $x \in R$

Let $\phi(x) = [x] + [1 - x]$, then

(a) ϕ is discontinuous somewhere on R

(b) ϕ is continuous on R but not differentiable only at $x = 1$

(c) ϕ is continuous on I but not differentiable at 1

(d) f is continuous on R but not differentiable at 1

5. For an arbitrary continuously differentiable function f, which of the following is a general solution of $z(px - qy) = y^2 - x^2$

(a) $x^2 - y^2 + z^2 = f(xy)$

(b) $(x + y)^2 + z^2 = f(xy)$

(c) $x^2 + y^2 + z^2 = f(y - x)$

(d) $x^2 - y^2 - z^2 = f((x - y)^2 + z^2)$

6. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables having an exponential distribution with mean $\frac{1}{\lambda}$.

Let $S_n = X_1 + X_2 + \ldots + X_n$ and $N = \inf\{n \ge 1 : S_n > 1\}$. Then, Var (N) equals

(a) 1 *(b)* λ *(c)* λ^2 *(d)* ∞

7. Let V be the space of twice differentiable functions on R satisfying $f'' - 2f' + f = 0$.

Define $T : V \to R^2$ by $T(f) = (f'(0), f(0))$. Then, T is

(a) one-to-one and onto

(b) one-to-one but not onto

(c) onto but not one-to-one

(d) neither one-to-one nor onto

8. The limit $\lim_{x\to 0} \frac{1}{x} \int_x^{2x} e^{-t^2}\, dt$

(a) does not exist

(b) is infinite

(c) exists and equal to 1

(d) exists and equal to 0

9. Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation

$$x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$$

are real, then

(a) $\lambda < \frac{4}{3}$ *(b)* $\lambda > \frac{5}{3}$

(c) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ *(d)* $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

10. If $P(x) = 51x^{101} - 2323x^{100} - 45x + 1035$, using Rolle's theorem atleast one root lies between interval

(a) $(45^{1/100}, 46)$

(b) $(44^{1/100}, 45)$

(c) $(46^{1/100}, 45)$

(d) $(45^{1/100}, 44)$

KNOWLEDGE Coefficient

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Q. No.	a.	b.	c.	d.	Q. No.	a.	b.	c.	d.
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2.	○	○	○	○	7.	○	○	○	○
3.	○	○	○	○	8.	○	○	○	○
4.	○	○	○	○	9.	○	○	○	○
5.	○	○	○	○	10.	○	○	○	○

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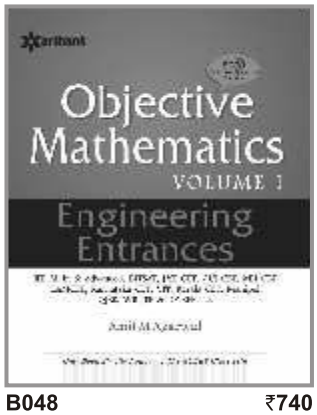

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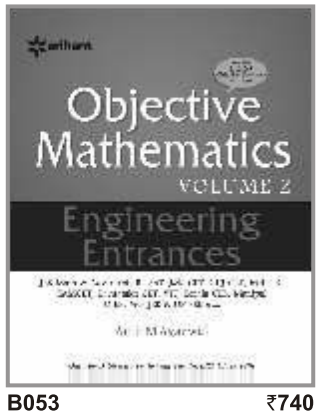

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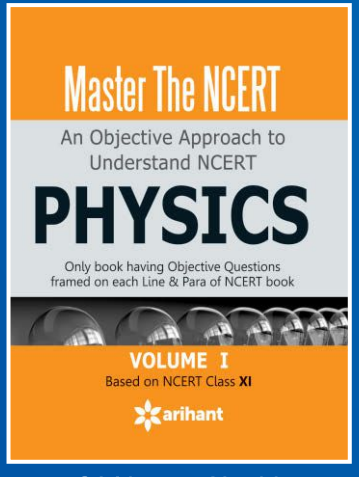
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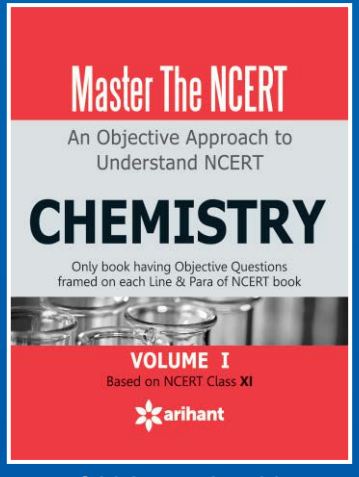
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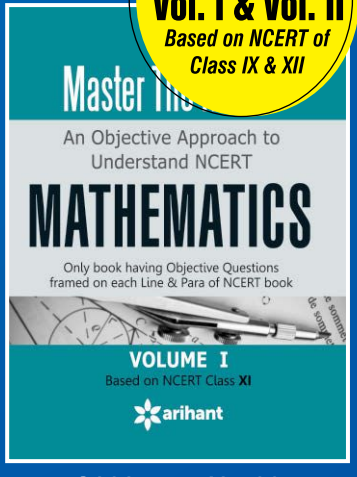
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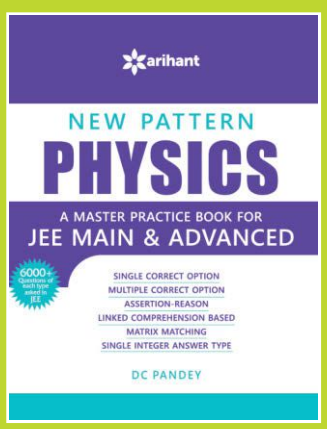

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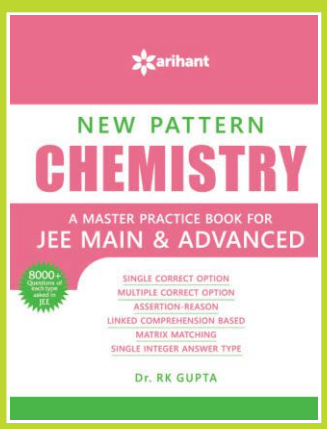

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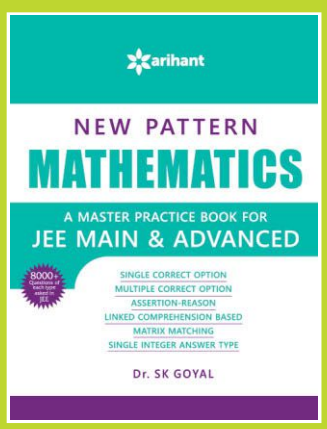

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